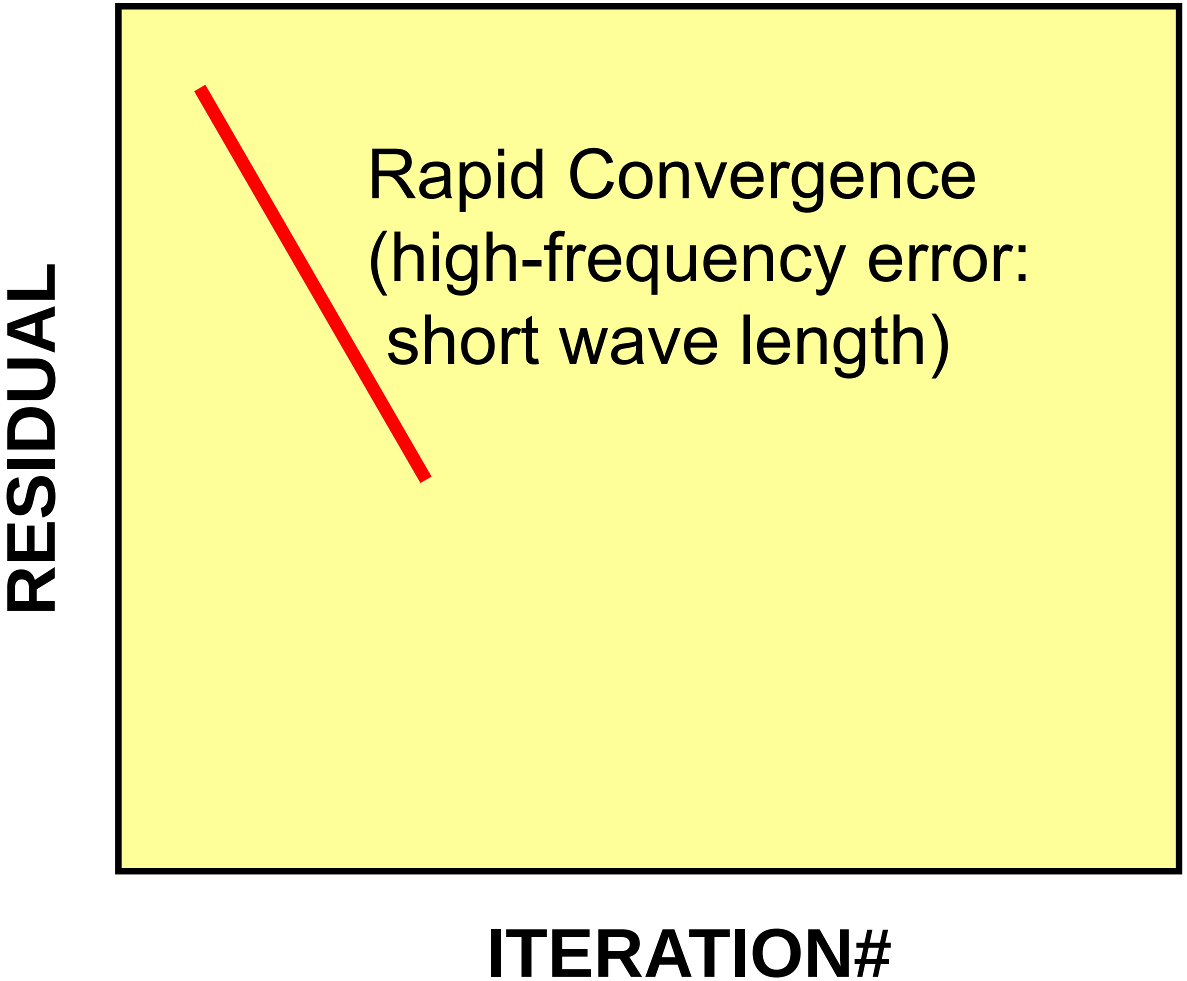
Around the multigrid in a single slide

- Multigrid is a scalable method for solving linear equations.
- Relaxation methods (smoother/smoothing operator in MG world) such as Gauss-Seidel efficiently damp high-frequency error but do not eliminate low-frequency error.
- The multigrid approach was developed in recognition that this low-frequency error can be accurately and efficiently solved on a coarser grid.
- Multigrid method uniformly damps all frequencies of error components with a computational cost that depends only linearly on the problem size (=scalable).
  - Good for large-scale computations
- Multigrid is also a good preconditioning algorithm for Krylov iterative solvers.

Convergence of Gauss-Seidel & SOR



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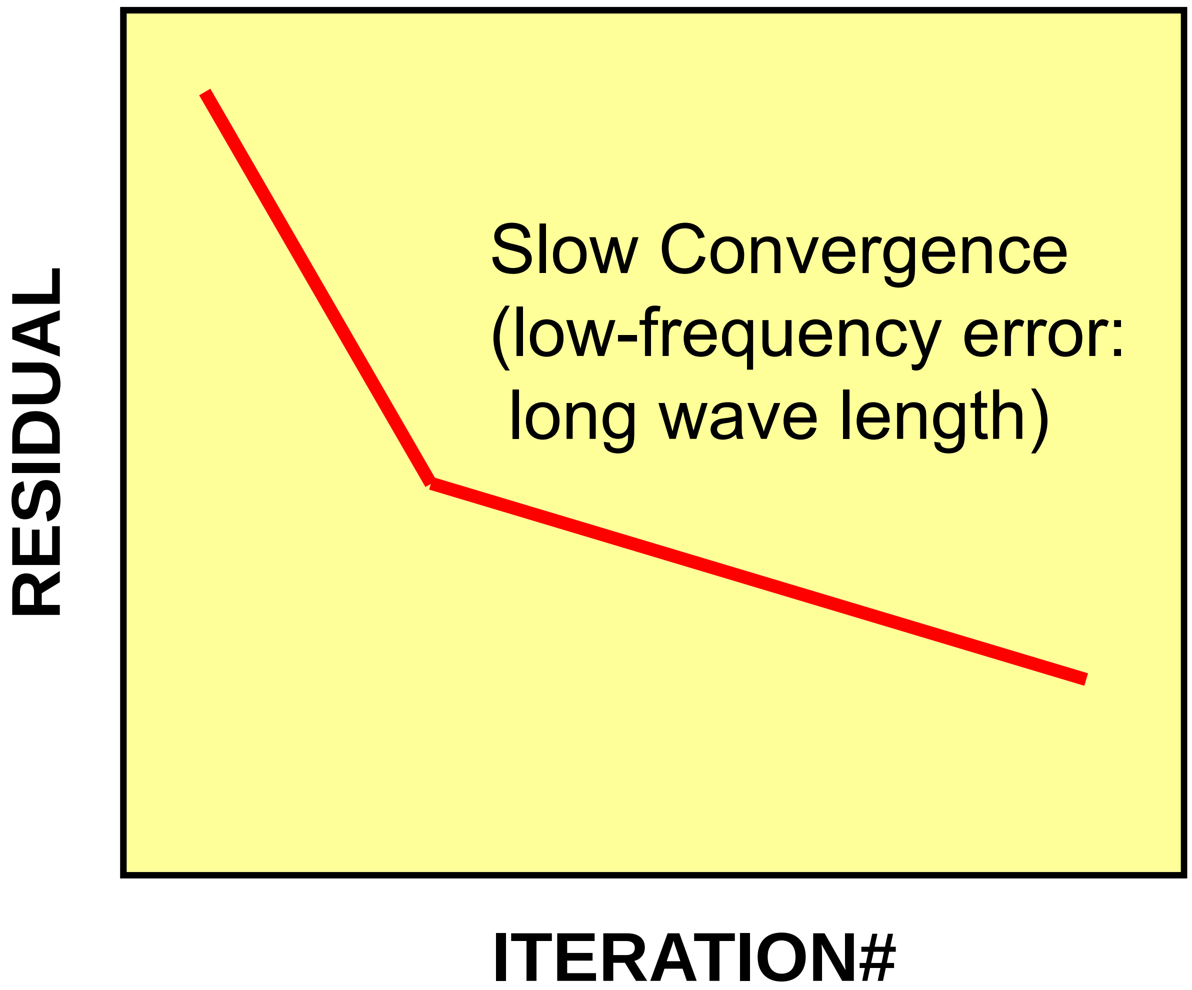

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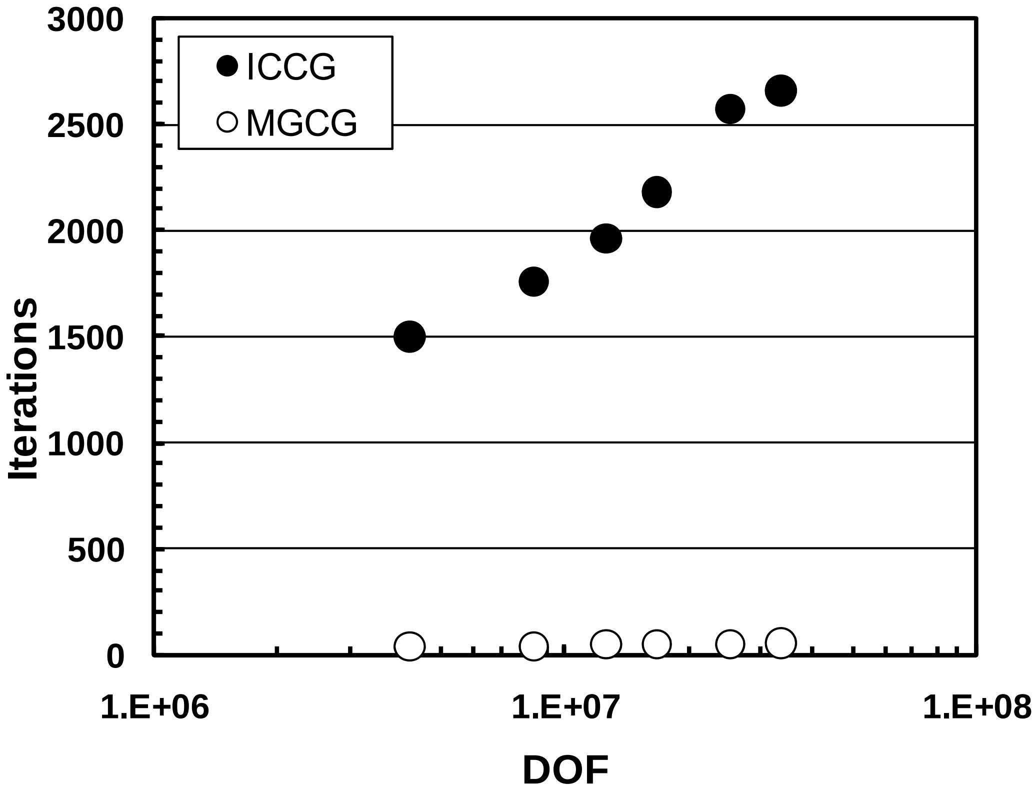
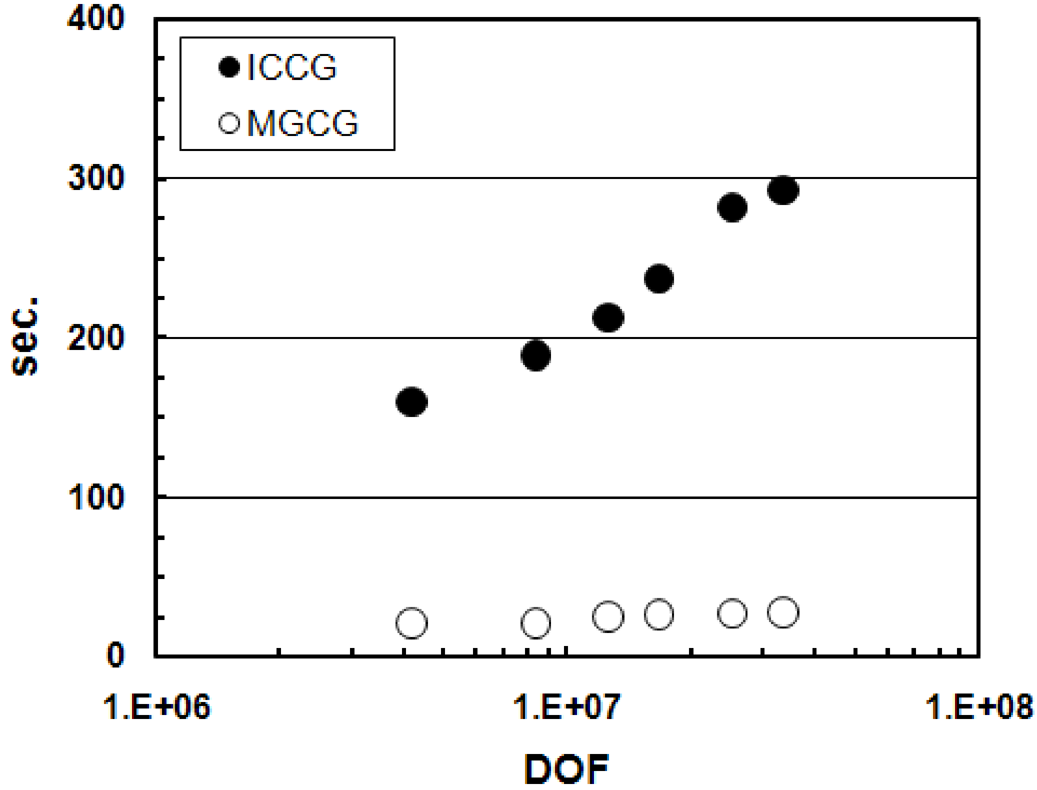
Multigrid is scalable
Weak Scaling: Problem Size/Core Fixed
for 3D Poisson Eqn's ($\Delta\phi = q$)

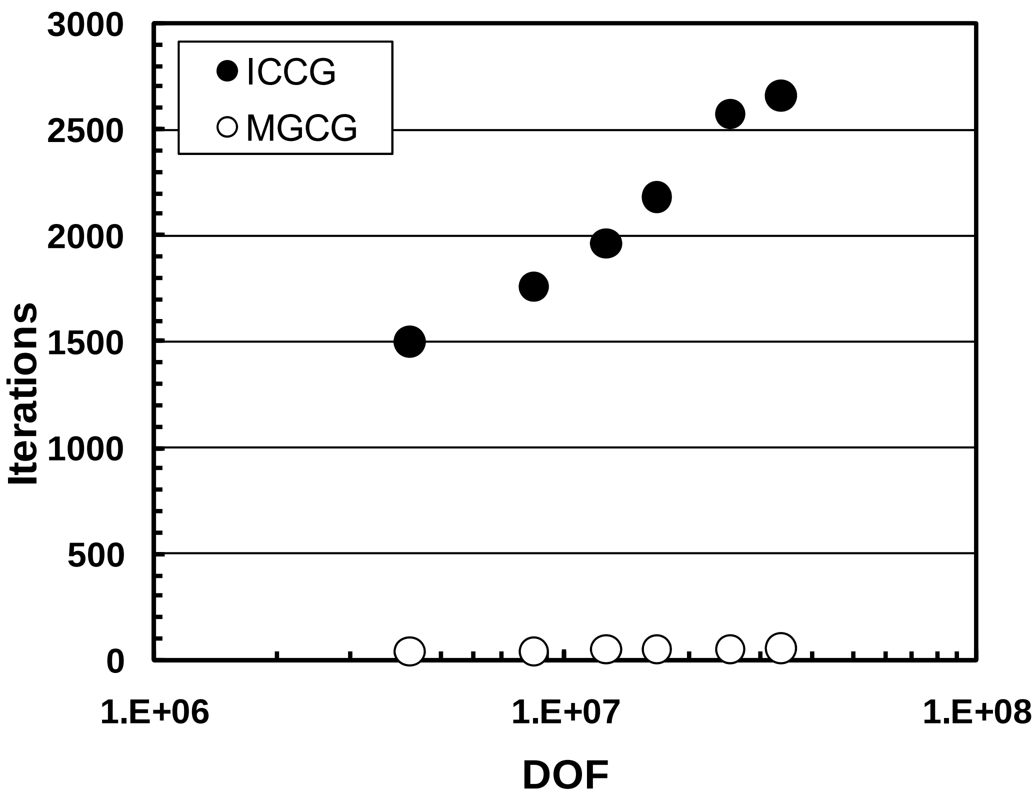
MGCG= Conjugate Gradient with Multigrid Preconditioning

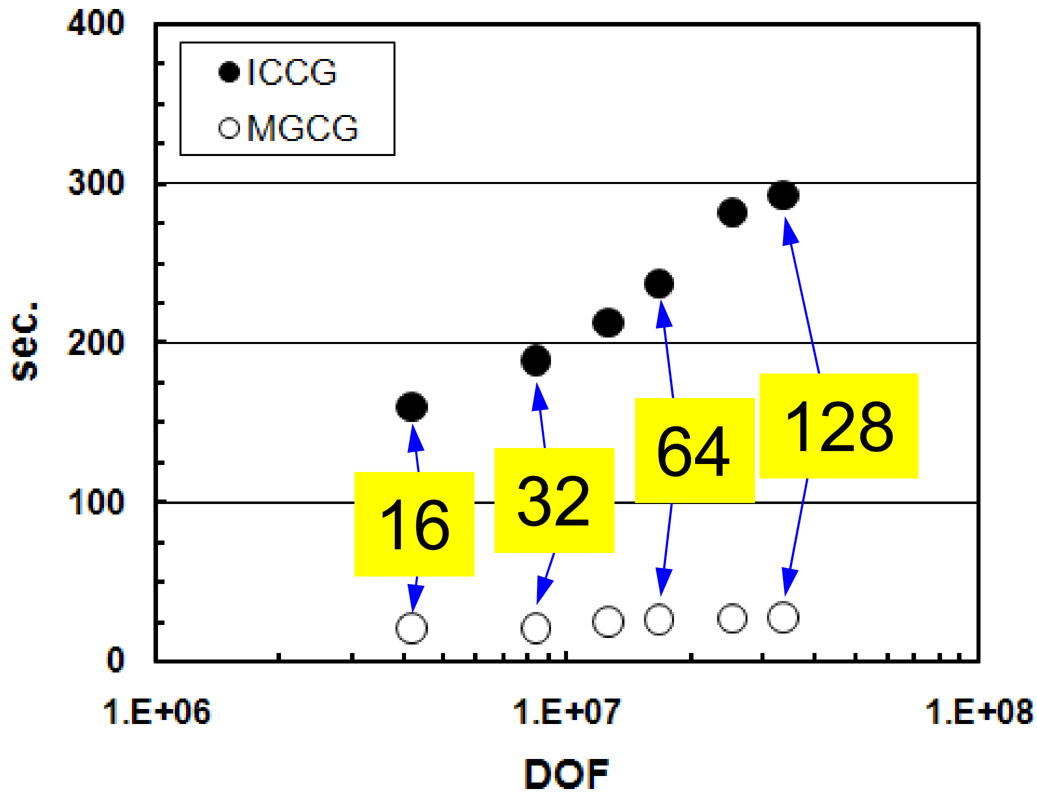




Multigrid is scalable
Weak Scaling: Problem Size/Core Fixed

Comp. time of MGCG for weak scaling is constant: => scalable





Procedure of Multigrid (1/3)

Multigrid is a scalable method for solving linear equations. Relaxation methods such as Gauss-Seidel efficiently damp high-frequency error but do not eliminate low-frequency error. The multigrid approach was developed in recognition that this low-frequency error can be accurately and efficiently solved on a coarser grid. This concept is explained here in the following simple 2-level method. If we have obtained the following linear system on a fine grid :

$$A_F \, u_F = f$$

and A_C as the discrete form of the operator on the coarse grid, a simple coarse grid correction can be given by :

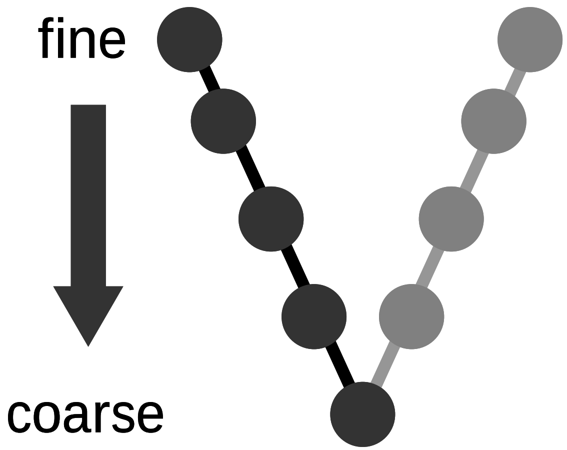
$$u_F^{(i+1)} = u_F^{(i)} + R^T A_C^{-1} R \, (f - A_F \, u_F^{(i)})$$

where R^T is the matrix representation of linear interpolation from the coarse grid to the fine grid (*prolongation* operator) and R is called the restriction operator. Thus, it is possible to calculate the residual on the fine grid, solve the coarse grid problem, and interpolate the coarse grid solution on the fine grid.

Procedure of Multigrid (2/3)

This process can be described as follows :

1. Relax the equations on the fine grid and obtain the result $u_F^{(i)} = S_F (A_F, f)$. This operator S_F (e.g., Gauss-Seidel) is called the *smoothing operator (or)*.
2. Calculate the residual term on the fine grid by $r_F = f - A_F u_F^{(i)}$.
3. *Restric*t the residual term on to the coarse grid by $r_C = R r_F$.
4. Solve the equation $A_C u_C = r_C$ on the coarse grid ; the accuracy of the solution on the coarse grid affects the convergence of the entire multigrid system.
5. Interpolate (or *prolong*) the coarse grid correction on the fine grid by $\mathrm{D}u_C^{(i)} = R^T u_C$.
6. Update the solution on the fine grid by $u_F^{(i+1)} = u_F^{(i)} + \mathrm{D}u_C^{(i)}$



$L^k W^k = F^k$ **(Linear Equation: Fine Level)**

$R^k = F^k - L^k w_1^k$

$v^k = W^k - w_1^k,\ L^k v^k = R^k$

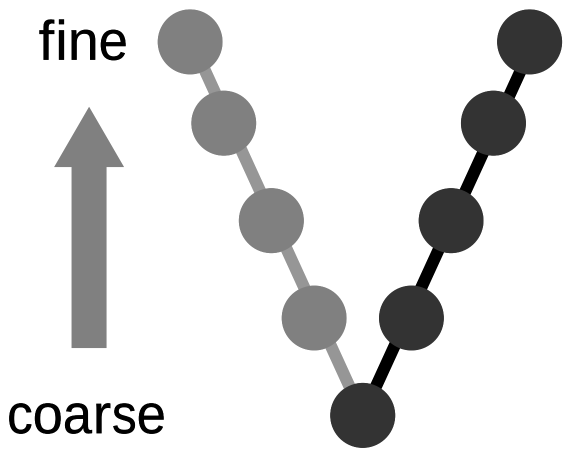
$R^{k-1} = I_k^{k-1} R^k$

$L^{k-1} v^{k-1} = R^{k-1}$ **(Linear Equation: Coarse Level)**

$v^k = I_{k-1}^k v^{k-1}$

$w_2^k = w_1^k + v^k$

w_1^k : Approx. Solution
v^k : Correction
I_k^{k-1} : Restriction Operator



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$L^{k-1} v^{k-1} = R^{k-1}$ **(Linear Equation: Coarse Level)**

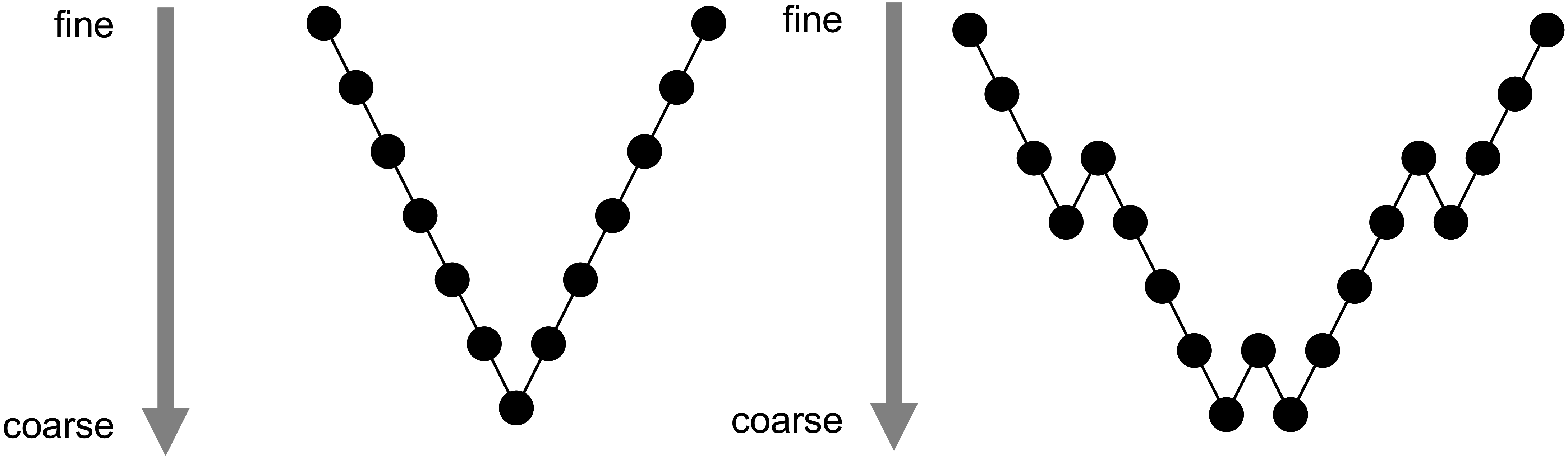
$v^k = I_{k-1}^k v^{k-1}$

$w_2^k = w_1^k + v^k$

I_{k-1}^k : Prolongation Operator
w_2^k : Approx. Solution by Multigrid

Procedure of Multigrid (3/3)

- Recursive application of this algorithm for 2-level procedure to consecutive systems of coarse-grid equations gives a multigrid V-cycle. If the components of the V-cycle are defined appropriately, the result is a method that uniformly damps all frequencies of error with a computational cost that depends only linearly on the problem size.
  - In other words, multigrid algorithms are *scalable*.
- In the V-cycle, starting with the finest grid, all subsequent coarser grids are visited only once.
  - In the down-cycle, smoothers damp oscillatory error components at different grid scales.
  - In the up-cycle, the smooth error components remaining on each grid level are corrected using the error approximations on the coarser grids.
- Alternatively, in a W-cycle, the coarser grids are solved more rigorously in order to reduce residuals as much as possible before going back to the more expensive finer grids.



(a) V-Cycle

(b) W-Cycle

Multigrid as a Preconditioner

- Multigrid algorithms tend to be problem-specific solutions and less robust than preconditioned Krylov iterative methods such as the IC/ILU methods.
- Fortunately, it is easy to combine the best features of multigrid and Krylov iterative methods into one algorithm
  - multigrid-preconditioned Krylov iterative methods.
- The resulting algorithm is robust, efficient and scalable.

- Mutigrid solvers and Krylov iterative solvers preconditioned by multigrid are intrinsically suitable for parallel computing.

Geometric and Algebraic Multigrid

- One of the most important issues in multigrid is the construction of the coarse grids.
- There are 2 basic multigrid approaches
  - geometric and algebraic
- In geometric multigrid, the geometry of the problem is used to define the various multigrid components.
- In contrast, algebraic multigrid methods use only the information available in the linear system of equations, such as matrix connectivity.
- Algebraic multigrid method (AMG) is suitable for applications with unstructured grids.
- Many tools for both geometric and algebraic methods on unstructured grids have been developed.

“Dark Side” of Multigrid Method

- Its performance is excellent for well-conditioned simple problems, such as homogeneous Poisson equations.
- But convergence could be worse for ill-conditioned problems.
- Extension of applicability of multigrid method is an active research area.

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