Comparison Index for Parallel Ordering in ILU Preconditioning Techniques

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Linear solvers

• Linear solvers are currently used in various kinds of numerical simulations.

  • Direct method

  • Iterative methods
    • To get an approximate solution with sufficient accuracy
    • Effective for linear systems of equations with a sparse coefficient matrix (e.g. FEM, FDM)
    • Advantages in memory and computational time
Iterative method

• Stationary iterative method
  • Jacobi, Gauss-Seidel, SOR

• Nonstationary iterative method
  • For symmetric coefficient matrix
    • CG, MINRES
  • For nonsymmetric coefficient matrix
    • Bi-CG, Bi-CGSTAB, GP-BiCG, GMRES
Preconditioning techniques

• Acceleration technique for iterative solver
• Essential for some applications

• Preconditioning
  • Transformation of $Ax=b$ into $M^{-1}Ax=M^{-1} b$
    • $M$: preconditioning matrix

• Many kinds of preconditioning techniques have been proposed
ILU preconditioning

• Classical, but powerful preconditioning
  – Most general preconditioning technique
• Effective in many practical areas
• Many variants and modified versions

• Parallel processing of ILU preconditioning is difficult due to forward and backward substitutions.
Parallel orderings

• One of popular ways for parallelization of ILU preconditioning

• In this technique, the unknown variables reordered to an appropriate form to parallel processing.

• In parallel ordering method, a trade-off problem between parallelism and convergence was found.
Research on parallel orderings (1)

By Duff and Meurant

Examination on convergence for various orderings

1. Trade-off problem between parallelism and convergence

Orderings having a higher degree of parallelism results in worse convergence.

2. The preconditioning effect is evaluated by the norm of the Remainder matrix $R$

\[ R = M - A \]

(M: Preconditioning matrix, A: Coefficient matrix)
Research on parallel orderings (2)
By Doi, Lichnewsky and Washio

• Introducing ordering graph
• Convergence evaluation index (Incompatibility ratio)
• Explanation on the trade-off problem
• Proposal of large-numbered multi-color ordering
Ordering graph

Ordering graph of red-black ordering

Direction: Order between neighbor nodes
Present research

• To propose a new comparison index for parallel ordering
  • The comparison index should be available in unstructured analyses.
    • The computational cost and memory requirement should be reduced compared with the Frobenius norm of the remainder matrix.
  • The comparison index includes the effect of non-incompatible nodes.
New evaluation index (S.R.I.)

Algorithm of ILU factorization, computation of remainder matrix, new evaluation index

\[
\begin{align*}
R &= O \\
I_{rp} &= 0 \\
&\text{for } I = 1 \text{ to } n-1 \\
&\quad \text{for } J = 1 \text{ to } n \\
&\quad \quad \text{for } K = 1 \text{ to } n \\
&\quad \quad \quad \text{if } a_{J,J} \neq 0 \& a_{I,K} \neq 0 \& a_{J,K} \neq 0 \quad \text{then} \\
&\quad \quad \quad \quad a_{J,K} \leftarrow a_{J,K} - a_{J,J} \cdot a_{I,K} / a_{I,I} \\
&\quad \quad \quad \quad \text{ILU factorization} \\
&\quad \quad \quad \text{endif} \\
&\quad \quad \quad \text{if } a_{J,J} \neq 0 \& a_{I,K} \neq 0 \& a_{J,K} = 0 \\
&\quad \quad \quad \quad r_{J,K} \leftarrow r_{J,K} + a_{J,J} \cdot a_{I,K} / a_{I,I} \\
&\quad \quad \quad \quad \text{Computation of remainder matrix} \\
&\quad \quad \quad \text{endif} \\
&\quad \text{Count of this update process} \\
&\text{endfor} \\
&\text{endfor} \\
&\text{endfor} \\
\end{align*}
\]

Evaluation index S.R.I. (Simple remainder index)
## S.R.I. in finite difference analysis

<table>
<thead>
<tr>
<th>Type</th>
<th>$l$</th>
<th>$l C_2$ (S.R.I./node)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type 0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Type 1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Type 2</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Type 3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>Type 4</td>
<td>4</td>
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<td>5</td>
<td>10</td>
</tr>
<tr>
<td>Type 6</td>
<td>6</td>
<td>15</td>
</tr>
</tbody>
</table>

S.R.I. in finite difference analysis

• Natural ordering has minimum S.R.I. value.
  • The average S.R.I. value per node in the natural ordering is given by 3.

• Example: Multi-color ordering
  • The average S.R.I. value per node is $3 + \frac{9}{m}$
    $m$: number of colors
Numerical tests

(1) Random orderings

Numerical results

Random ordering tests

- Incompatible node ratio
- Simple remainder index

#iteration

Graphs showing the relationship between iteration and incompatibility ratios and simple remainder indices.
# Evaluation by S.R.I.

## Natural ordering

<table>
<thead>
<tr>
<th># Ite.</th>
<th>S.R.I.</th>
</tr>
</thead>
<tbody>
<tr>
<td>169</td>
<td>2.954</td>
</tr>
</tbody>
</table>

## Multi-color ordering

<table>
<thead>
<tr>
<th># Colors</th>
<th># Ite.</th>
<th>S.R.I.</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>266</td>
<td>7.384</td>
</tr>
<tr>
<td>4</td>
<td>225</td>
<td>5.169</td>
</tr>
<tr>
<td>8</td>
<td>204</td>
<td>4.061</td>
</tr>
<tr>
<td>16</td>
<td>192</td>
<td>3.507</td>
</tr>
<tr>
<td>32</td>
<td>187</td>
<td>3.231</td>
</tr>
<tr>
<td>64</td>
<td>188</td>
<td>3.092</td>
</tr>
<tr>
<td>128</td>
<td>184</td>
<td>3.023</td>
</tr>
</tbody>
</table>

## Block red-black ordering

<table>
<thead>
<tr>
<th>Block size</th>
<th># Ite.</th>
<th>S.R.I.</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>207</td>
<td>4.402</td>
</tr>
<tr>
<td>4</td>
<td>184</td>
<td>3.487</td>
</tr>
<tr>
<td>8</td>
<td>183</td>
<td>3.169</td>
</tr>
<tr>
<td>16</td>
<td>180</td>
<td>3.044</td>
</tr>
<tr>
<td>32</td>
<td>180</td>
<td>2.990</td>
</tr>
<tr>
<td>64</td>
<td>169</td>
<td>2.965</td>
</tr>
</tbody>
</table>

Relationship between the number of iteration and Simple remainder index
New evaluation index

Algorithm of ILU factorization, computation of remainder matrix, new evaluation index

\[ \mathbf{R} = \mathbf{O} \]
\[ I_{rp} = 0 \]

\[ \text{for } I = 1 \text{ to } n-1 \]
\[ \text{for } J = 1 \text{ to } n \]
\[ \text{for } K = 1 \text{ to } n \]
\[ \text{if } a_{j,1} \neq 0 \& a_{1,K} \neq 0 \& a_{j,K} \neq 0 \text{ then} \]
\[ a_{j,K} \leftarrow a_{j,K} - a_{j,1} \cdot a_{1,K} / a_{1,1} \]  
ILU factorization
\[ \text{endif} \]
\[ \text{if } a_{j,1} \neq 0 \& a_{1,K} \neq 0 \& a_{j,K} = 0 \text{ then} \]
\[ r_{j,K} \leftarrow r_{j,K} + a_{j,1} \cdot a_{1,K} / a_{1,1} \]  
Computation of remainder matrix
\[ I_{rp} \leftarrow I_{rp} + |a_{j,1} \cdot a_{1,K} / a_{1,1}| \]  
Computation of P.R.I.
\[ \text{endif} \]
\[ \text{endfor} \]
\[ \text{endfor} \]
\[ \text{endfor} \]

Evaluation index

P.R.I.

(Precise remainder index)
Relationship between P.R.I. and Remainder matrix

When

• Coefficient matrix is symmetric, and
• all its diagonal entries have the same sign,

$$P.R.I. = ||R||_A$$

For general coefficient matrix, it holds that

$$||R||_F \leq P.R.I.$$
P.R.I. for variants of ILU preconditioning (1)

(1) ILU factorization applied to modified coefficient matrix

- Shifted ICCG method
  - Preconditioner matrix is forced to be positive-definite
- Acceleration technique for convergence

\[
P.R.I. = I_{rp0} + || \Delta A ||_A
\]

- \( I_{rp0} \): Summation of absolute values of dropped fill-ins
- \( \Delta A \): Modification term
P.R.I. for variants of ILU preconditioning (2)

(2) ILU(l) preconditioning

- Level-l fill-ins are allowed

P.R.I. is determined by the algorithm for P.R.I. for ILU(0) preconditioning.
Numerical tests

• ILU preconditioning for symmetric coefficient matrixes – Incomplete Cholesky preconditioning

• ICCG method (conjugate gradient method)

• Use of 51 random orderings
  – Unknowns (0% - 100% by 2%) are permutated at random from original data.

• Original ordering is similar to natural ordering.

• Computer: Fujitsu HPC2500 (SPARC-compatible processors)

• Convergence criteria: Relative residual norm < $10^{-7}$
Numerical tests

(1) Matrix Market data  S1RMQ4M1  
    - Structure analysis with shell elements

(2) Matrix Market data  S3RMT3M1  
    - Structure analysis with a triangular mesh

(3) 2-D finite difference analysis of Poisson equation

(4) 3-D electromagnetic field analysis  
    - Finite edge-element analysis
Numerical results

(Matrix market data 1)

Relationship between P.R.I. and number of iterations

Convergence behavior

Good estimation of convergence

P.R.I. is effective for evaluation of orderings.

Correlation coefficient: 0.86 High!
Numerical results
(Matrix market data 2)

General characteristics can be predicted.

Convergence behavior
There are many spikes in the convergence behavior. The estimation of convergence is difficult.

Correlation coefficient: 0.81
Numerical results
(2-D finite difference analysis)

Dirichlet boundary problem of Poisson equation

\[- \nabla \cdot (k \nabla u(x, y)) = f \text{ in } \Omega (0,1) \times (0,1)\]
\[u(x, y) = 0 \text{ on } \partial \Omega\]
\[if \ (1/4 \leq x \leq 3/4 \text{ and } 1/4 \leq y \leq 3/4) \text{ then} \]
\[k = 100 \text{ else } k = 1\]

- 100 x 100 mesh

Relationship between P.R.I. and number of iterations

Good estimation is done.
Numerical results
(3-D electromagnetic filed analysis)

Model: IEEJ standard benchmark model of 3-D eddy current analysis

<table>
<thead>
<tr>
<th>Number of elements</th>
<th>327680</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of nodes</td>
<td>342225</td>
</tr>
<tr>
<td>Number of DOFs</td>
<td>1011920</td>
</tr>
</tbody>
</table>

Basic equation
\[
\nabla \times (\nu \nabla \times \mathbf{A}) = -\sigma \frac{\partial \mathbf{A}}{\partial t} + \mathbf{J}_0
\]

A-Method (A: magnetic vector potential)

Finite edge-element method
Numerical results
(3-D electromagnetic filed analysis)

Finite edge-element analysis of electromagnetic field problem

The coefficient matrix is indefinite.

In this analysis (low-frequency analysis), the coefficient matrix is semi-positive definite.

The shifted ICCG is used.

The shift parameter is given by 0.3.
Numerical results
(3-D eddy-current analysis)

Relationship between P.R.I. and number of iterations

Two tests were performed.

General characteristics is estimated.

Some errors were detected.

Convergence behavior

Convergence behavior line is smooth.

No big spike.
Numerical tests show a strong correlation between P.R.I. and the Frobenius norm of the remainder matrix.

The additional memory requirement for computing P.R.I. is much smaller than that for a Frobenius norm of a remainder matrix.
Conclusions

• We proposed a new evaluation index of orderings in ILU preconditioning, S.R.I. (Simple Remainder Index) and P.R.I. (Precise Remainder Index).

• Numerical tests using coefficient matrices downloaded from the Matrix Market, a 2-D finite difference analysis of Poisson equation, and a 3-D electromagnetic field simulation confirms the effectiveness of the evaluation index.
Conclusions

• The evaluation index P.R.I. has a strong relationship with the Frobenius norm of the remainder matrix, and its memory cost is trivial.

• (Future work) The evaluation index P.R.I. can be used for examining effects of parameters introduced in variants of ILU preconditioning technique. We would confirm the effectiveness of the P.R.I. in such a case.