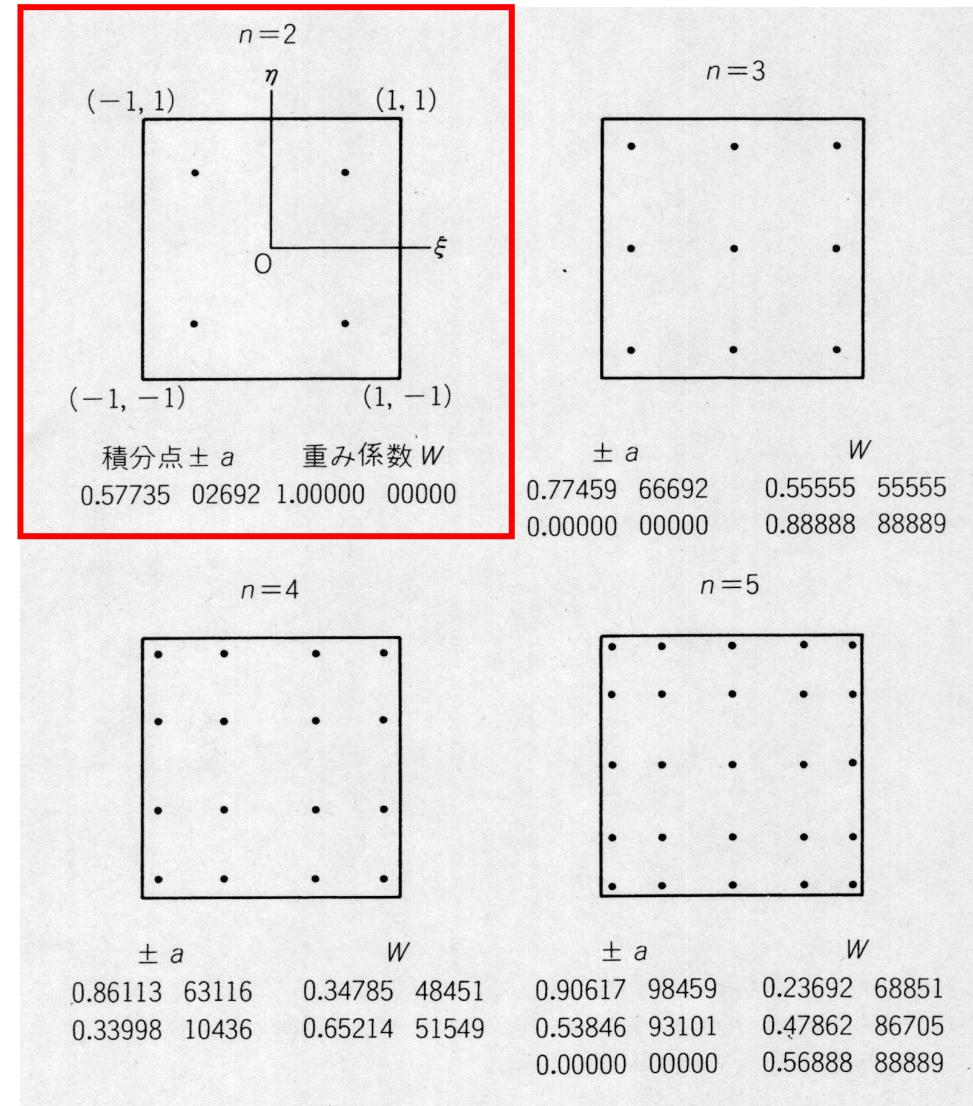


ガウスの積分公式

Gaussian quadrature

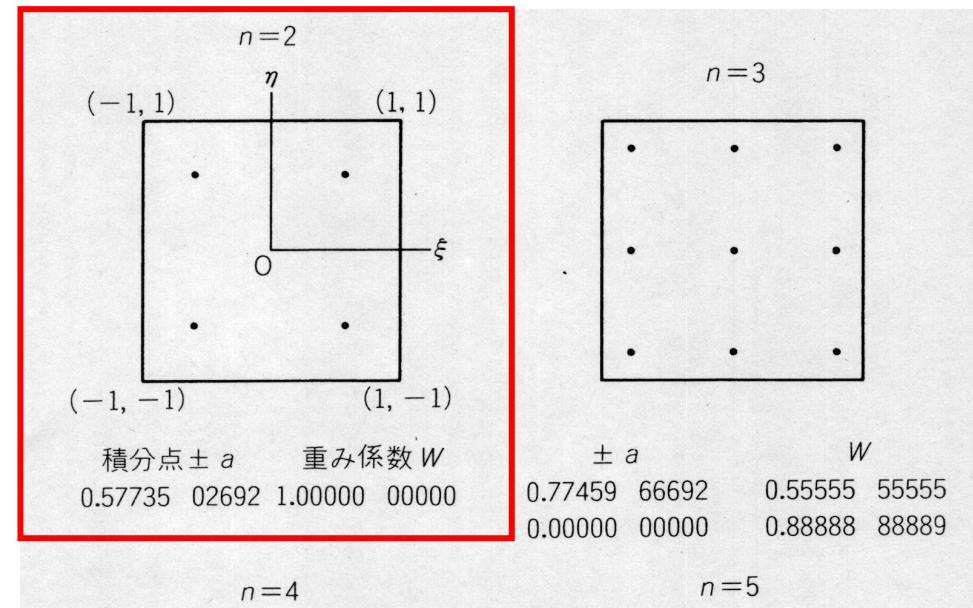
この組み合わせがよく使用される。4点における f の値を使用して領域における積分の値を近似する。



ガウスの積分公式

Gaussian quadrature

この組み合わせがよく使用される。4点における f の値を使用して領域における積分の値を近似する。

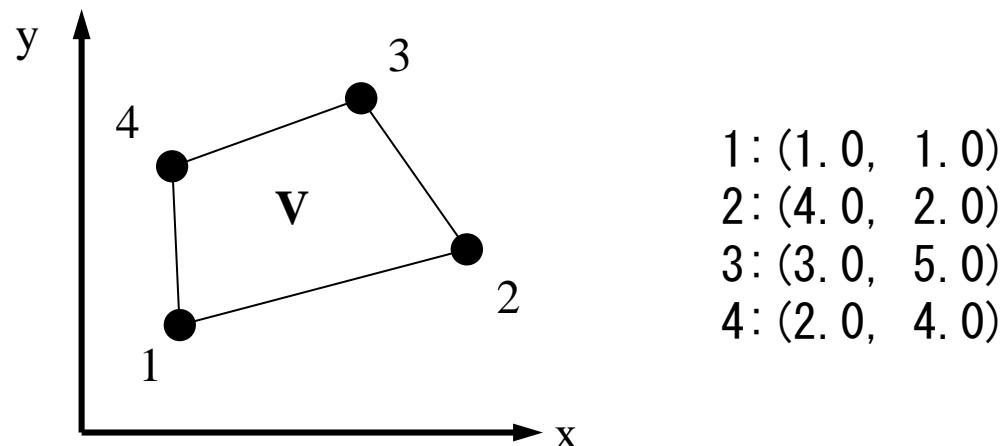


$$\begin{aligned}
 I &= \int_{-1}^{+1} \int_{-1}^{+1} f(\xi, \eta) d\xi d\eta = \sum_{i=1}^m \sum_{j=1}^n [W_i \cdot W_j \cdot f(\xi_i, \eta_j)] \\
 &= 1.0 \times 1.0 \times f(-0.57735, -0.57735) + 1.0 \times 1.0 \times f(-0.57735, +0.57735) \\
 &\quad + 1.0 \times 1.0 \times f(+0.57735, +0.57735) + 1.0 \times 1.0 \times f(+0.57735, -0.57735)
 \end{aligned}$$

0.33998	1.0436	0.05214	0.1549	0.05840	0.95101	0.47882	0.00705
0.00000	0.00000	0.56888	88889				

宿題

- ガウスの積分公式を使用して以下の四角形の面積を求めよ



$$I = \int_V dV$$

ヒント

- 座標値によってヤコビアン（ヤコビの行列）を計算する。
- ガウスの積分公式（n=2）に代入する。

$$I = \int_{-1}^{+1} \int_{-1}^{+1} f(\xi, \eta) d\xi d\eta = \sum_{i=1}^m \sum_{j=1}^n [W_i \cdot W_j \cdot f(\xi_i, \eta_j)]$$

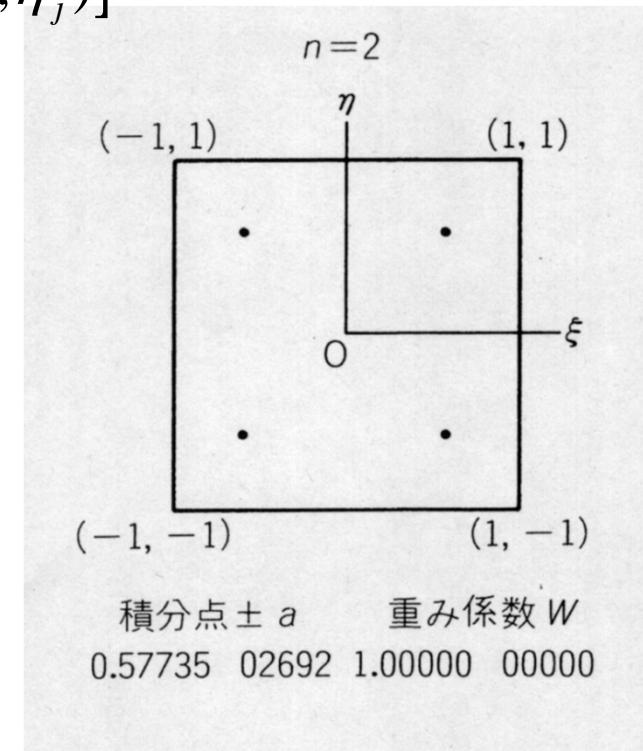
```

implicit REAL*8 (A-H,O-Z)
real*8 W(2)
real*8 POI(2)

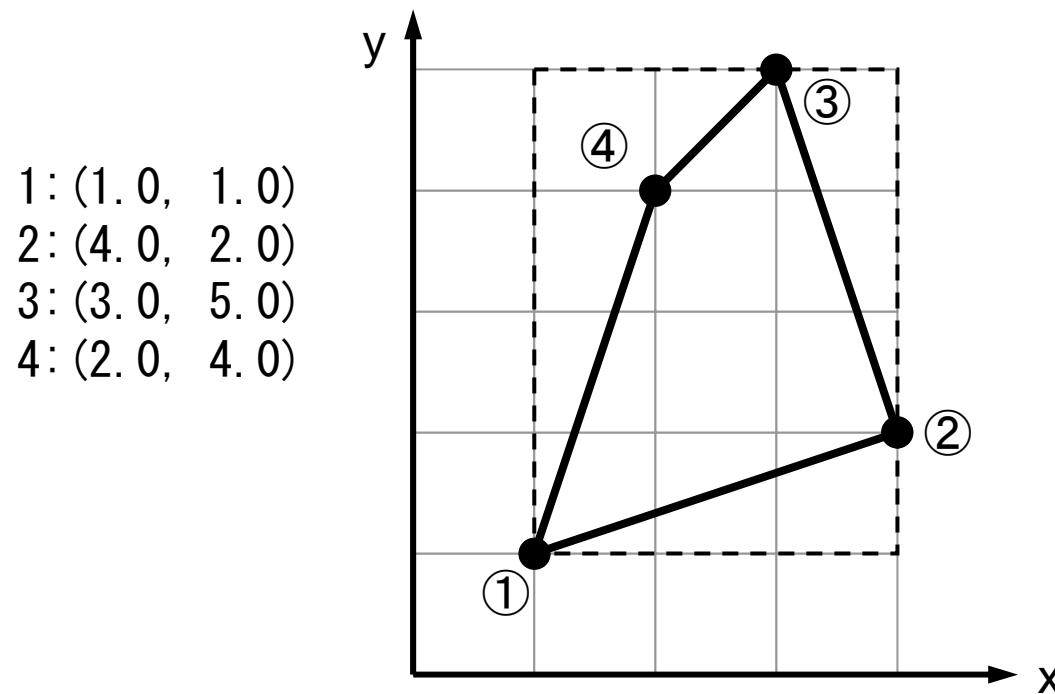
W(1)= 1.0d0
W(2)= 1.0d0
POI(1)= -0.5773502692d0
POI(2)= +0.5773502692d0

SUM= 0.d0
do jp= 1, 2
do ip= 1, 2
  FC = F(POI(ip),POI(jp))
  SUM= SUM + W(ip)*W(jp)*FC
enddo
enddo

```



正解



[] から四隅の三角形を取り除く

$$\begin{aligned} & 3 \times 4 - \frac{1}{2}(3+3+2+4) \times 1 \\ & = 12 - \frac{12}{2} = 6 \end{aligned}$$

何をすべきか?

- 以下の積分を求めればよい：

$$I = \int_V dV = \iint dx dy = \int_{-1}^{+1} \int_{-1}^{+1} \det[J] d\xi d\eta$$

- 定義によれば：

$$I = \int_{-1}^{+1} \int_{-1}^{+1} f(\xi, \eta) d\xi d\eta = \sum_{i=1}^m \sum_{j=1}^n [W_i \cdot W_j \cdot f(\xi_i, \eta_j)]$$

- つまり, $\det[J] = f$
- ということは $\det[J]$ の積分点における値を求めれば良い！

$$\det[J(\xi_i, \eta_j)]$$

初期設定 (1/4)

```
implicit REAL*8 (A-H,O-Z)

real*8 X(4), Y(4)
real*8 W(2), POS(2)
real*8 SHAPE(2,2,4)
real*8 PNQ(2,4), PNE(2,4), DETJ(2,2)
```

```
!C
!C-- POINT data
X(1)= 1.0
Y(1)= 1.0

X(2)= 4.0
Y(2)= 2.0

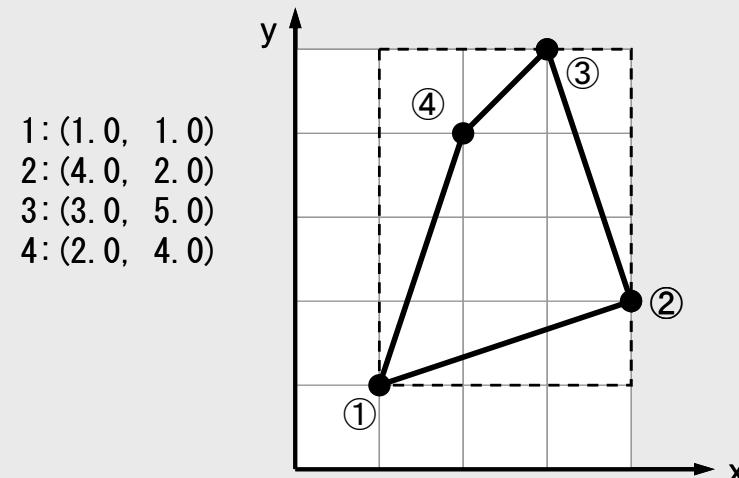
X(3)= 3.0
Y(3)= 5.0

X(4)= 2.0
Y(4)= 4.0
```

```
!C
!C-- Quadrature points & weighting coef.
W(1)= +1.0000000000d+00
W(2)= +1.0000000000d+00
```

```
POS(1)= -0.5773502692d+00
POS(2)= +0.5773502692d+00
```

各節点の座標



初期設定 (1/4)

```
implicit REAL*8 (A-H,O-Z)

real*8 X(4), Y(4)
real*8 W(2), POS(2)
real*8 SHAPE(2,2,4)
real*8 PNQ(2,4), PNE(2,4), DETJ(2,2)
```

```
!C
!C-- POINT data
X(1)= 1.0
Y(1)= 1.0
```

```
X(2)= 4.0
Y(2)= 2.0
```

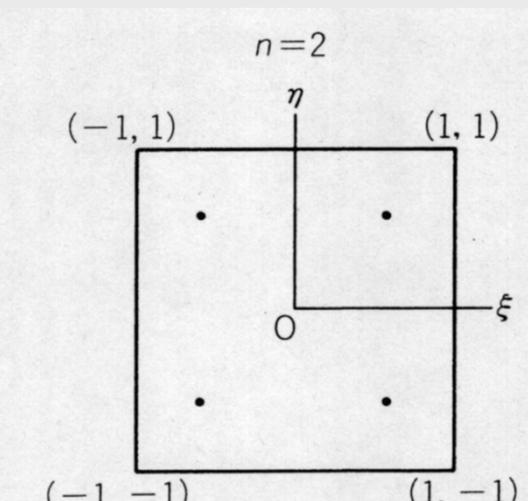
```
X(3)= 3.0
Y(3)= 5.0
```

```
X(4)= 2.0
Y(4)= 4.0
```

```
!C
!C-- Quadrature points & weighting coef.
W(1)= +1.0000000000d+00
W(2)= +1.0000000000d+00
```

```
POS(1)= -0.5773502692d+00
POS(2)= +0.5773502692d+00
```

POS: 積分点座標
W: 重み係数



積分点 a 重み係数 W
0.57735 02692 1.00000 00000

積分点における形状関数, その微分 (2/4)

```

!C
!C-- SHAPE functions
O4th= 0.25d0

do jp= 1, 2
do ip= 1, 2
  QP1= 1.d0 + POS(ip)
  QM1= 1.d0 - POS(ip)
  EP1= 1.d0 + POS(jp)
  EM1= 1.d0 - POS(jp)

  SHAPE(ip,jp,1)= O4th * QM1 * EM1
  SHAPE(ip,jp,2)= O4th * QP1 * EM1
  SHAPE(ip,jp,3)= O4th * QP1 * EP1
  SHAPE(ip,jp,4)= O4th * QM1 * EP1

  PNQ(jp,1)= - O4th * EM1
  PNQ(jp,2)= + O4th * EM1
  PNQ(jp,3)= + O4th * EP1
  PNQ(jp,4)= - O4th * EP1

  PNE(ip,1)= - O4th * QM1
  PNE(ip,2)= - O4th * QP1
  PNE(ip,3)= + O4th * QP1
  PNE(ip,4)= + O4th * QM1
enddo
enddo

```

$$\begin{aligned}
QP1(i) &= (1 + \xi_i), & QM1(i) &= (1 - \xi_i) \\
EP1(j) &= (1 + \eta_j), & EM1(j) &= (1 - \eta_j)
\end{aligned}$$

積分点における形状関数, その微分 (2/4)

```

!C
!C-- SHAPE functions
O4th= 0.25d0

do jp= 1, 2
do ip= 1, 2
  QP1= 1.d0 + POS(ip)
  QM1= 1.d0 - POS(ip)
  EP1= 1.d0 + POS(jp)
  EM1= 1.d0 - POS(jp)

  SHAPE(ip, jp, 1)= O4th * QM1 * EM1
  SHAPE(ip, jp, 2)= O4th * QP1 * EM1
  SHAPE(ip, jp, 3)= O4th * QP1 * EP1
  SHAPE(ip, jp, 4)= O4th * QM1 * EP1

  PNQ(jp, 1)= - O4th * EM1
  PNQ(jp, 2)= + O4th * EM1
  PNQ(jp, 3)= + O4th * EP1
  PNQ(jp, 4)= - O4th * EP1

  PNE(ip, 1)= - O4th * QM1
  PNE(ip, 2)= - O4th * QP1
  PNE(ip, 3)= + O4th * QP1
  PNE(ip, 4)= + O4th * QM1

enddo
enddo

```

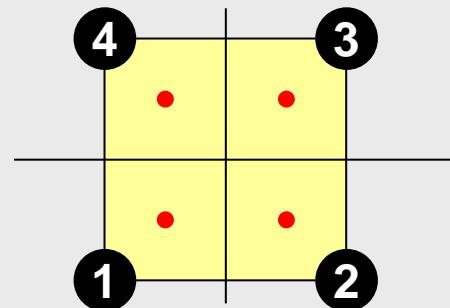
SHAPE: (ξ_i, η_j) における形状関数の値

$$N_1(\xi_i, \eta_j) = \frac{1}{4}(1 - \xi_i)(1 - \eta_j)$$

$$N_2(\xi_i, \eta_j) = \frac{1}{4}(1 + \xi_i)(1 - \eta_j)$$

$$N_3(\xi_i, \eta_j) = \frac{1}{4}(1 + \xi_i)(1 + \eta_j)$$

$$N_4(\xi_i, \eta_j) = \frac{1}{4}(1 - \xi_i)(1 + \eta_j)$$



積分点における形状関数, その微分 (2/4)

```

!C
!C-- SHAPE functions
O4th= 0.25d0

do jp= 1, 2
do ip= 1, 2
  QP1= 1.d0 + POS(ip)
  QM1= 1.d0 - POS(ip)
  EP1= 1.d0 + POS(jp)
  EM1= 1.d0 - POS(jp)

  SHAPE(ip, jp, 1)= O4th * QM1 * EM1
  SHAPE(ip, jp, 2)= O4th * QP1 * EM1
  SHAPE(ip, jp, 3)= O4th * QP1 * EP1
  SHAPE(ip, jp, 4)= O4th * QM1 * EP1

  PNQ(jp, 1)= - O4th * EM1
  PNQ(jp, 2)= + O4th * EM1
  PNQ(jp, 3)= + O4th * EP1
  PNQ(jp, 4)= - O4th * EP1

  PNE(ip, 1)= - O4th * QM1
  PNE(ip, 2)= - O4th * QP1
  PNE(ip, 3)= + O4th * QP1
  PNE(ip, 4)= + O4th * QM1

enddo
enddo

```

$$PNQ(j, k) = \frac{\partial N_k}{\partial \xi} (\xi = \xi_i, \eta = \eta_j)$$

$$PNE(j, k) = \frac{\partial N_k}{\partial \eta} (\xi = \xi_i, \eta = \eta_j)$$

$$\frac{\partial N_1}{\partial \xi}(\xi_i, \eta_j) = -\frac{1}{4}(1 - \eta_j) \quad \frac{\partial N_1}{\partial \eta}(\xi_i, \eta_j) = -\frac{1}{4}(1 - \xi_i)$$

$$\frac{\partial N_2}{\partial \xi}(\xi_i, \eta_j) = +\frac{1}{4}(1 - \eta_j) \quad \frac{\partial N_2}{\partial \eta}(\xi_i, \eta_j) = -\frac{1}{4}(1 + \xi_i)$$

$$\frac{\partial N_3}{\partial \xi}(\xi_i, \eta_j) = +\frac{1}{4}(1 + \eta_j) \quad \frac{\partial N_3}{\partial \eta}(\xi_i, \eta_j) = +\frac{1}{4}(1 + \xi_i)$$

$$\frac{\partial N_4}{\partial \xi}(\xi_i, \eta_j) = -\frac{1}{4}(1 + \eta_j) \quad \frac{\partial N_4}{\partial \eta}(\xi_i, \eta_j) = +\frac{1}{4}(1 - \xi_i)$$

(ξ_i, η_j) における形状関数の一階微分

積分点におけるヤコビアン 計算 (3/4)

```

!C
!C +-----+
!C | JACOBIAN matrix |
!C +-----+
!C===
      do jp= 1, 2
      do ip= 1, 2
        dXdQ = PΝQ(jp,1)*X(1) + PΝQ(jp,2)*X(2) +
&          PΝQ(jp,3)*X(3) + PΝQ(jp,4)*X(4)
        dYdQ = PΝQ(jp,1)*Y(1) + PΝQ(jp,2)*Y(2) +
&          PΝQ(jp,3)*Y(3) + PΝQ(jp,4)*Y(4)
        dXdE = PΝE(ip,1)*X(1) + PΝE(ip,2)*X(2) +
&          PΝE(ip,3)*X(3) + PΝE(ip,4)*X(4)
        dYdE = PΝE(ip,1)*Y(1) + PΝE(ip,2)*Y(2) +
&          PΝE(ip,3)*Y(3) + PΝE(ip,4)*Y(4)
        DETJ(ip,jp)= dXdQ*dYdE - dXdE*dYdQ
      enddo
    enddo
!C===

```

$$\begin{aligned}
dXdQ &= \frac{\partial x}{\partial \xi} & dYdQ &= \frac{\partial y}{\partial \xi} \\
dXdE &= \frac{\partial x}{\partial \eta} & dYdE &= \frac{\partial y}{\partial \eta} \\
DETJ(i, j) &= \det[J(\xi_i, \eta_j)]
\end{aligned}$$

$$J_{11} = \frac{\partial x}{\partial \xi} = \frac{\partial}{\partial \xi} \left(\sum_{i=1}^4 N_i x_i \right) = \sum_{i=1}^4 \frac{\partial N_i}{\partial \xi} x_i, \quad J_{12} = \frac{\partial y}{\partial \xi} = \frac{\partial}{\partial \xi} \left(\sum_{i=1}^4 N_i y_i \right) = \sum_{i=1}^4 \frac{\partial N_i}{\partial \xi} y_i,$$

$$J_{21} = \frac{\partial x}{\partial \eta} = \frac{\partial}{\partial \eta} \left(\sum_{i=1}^4 N_i x_i \right) = \sum_{i=1}^4 \frac{\partial N_i}{\partial \eta} x_i, \quad J_{22} = \frac{\partial y}{\partial \eta} = \frac{\partial}{\partial \eta} \left(\sum_{i=1}^4 N_i y_i \right) = \sum_{i=1}^4 \frac{\partial N_i}{\partial \eta} y_i$$

数値積分実施 (4/4)

```

!C
!C +-----+
!C | AREA |
!C +-----+
!C ===

      AREA= 0.d0
      do jp= 1, 2
      do ip= 1, 2
          AREA= AREA + dabs(DETJ(ip, jp)) * W(ip) * W(jp)
      enddo
      enddo

!C
!C-- ANALYTICAL SOLUTION
      XA2= X(2) - X(1)
      YA2= Y(2) - Y(1)
      XA3= X(3) - X(1)
      YA3= Y(3) - Y(1)
      XA4= X(4) - X(1)
      YA4= Y(4) - Y(1)

      AREAA= 0.50d0 * (dabs(XA2*YA3-YA2*XA3) +dabs(XA3*YA4-YA3*XA4))

!C ===

      write (*,'(a,1pe16.6)') 'Gaussian quadrature', AREA
      write (*,'(a,1pe16.6)') 'analytical sol. ', AREAA

      stop
      end

```

$$I = \int_{-1}^{+1} \int_{-1}^{+1} f(\xi, \eta) d\xi d\eta = \sum_{i=1}^m \sum_{j=1}^n [W_i \cdot W_j \cdot f(\xi_i, \eta_j)]$$