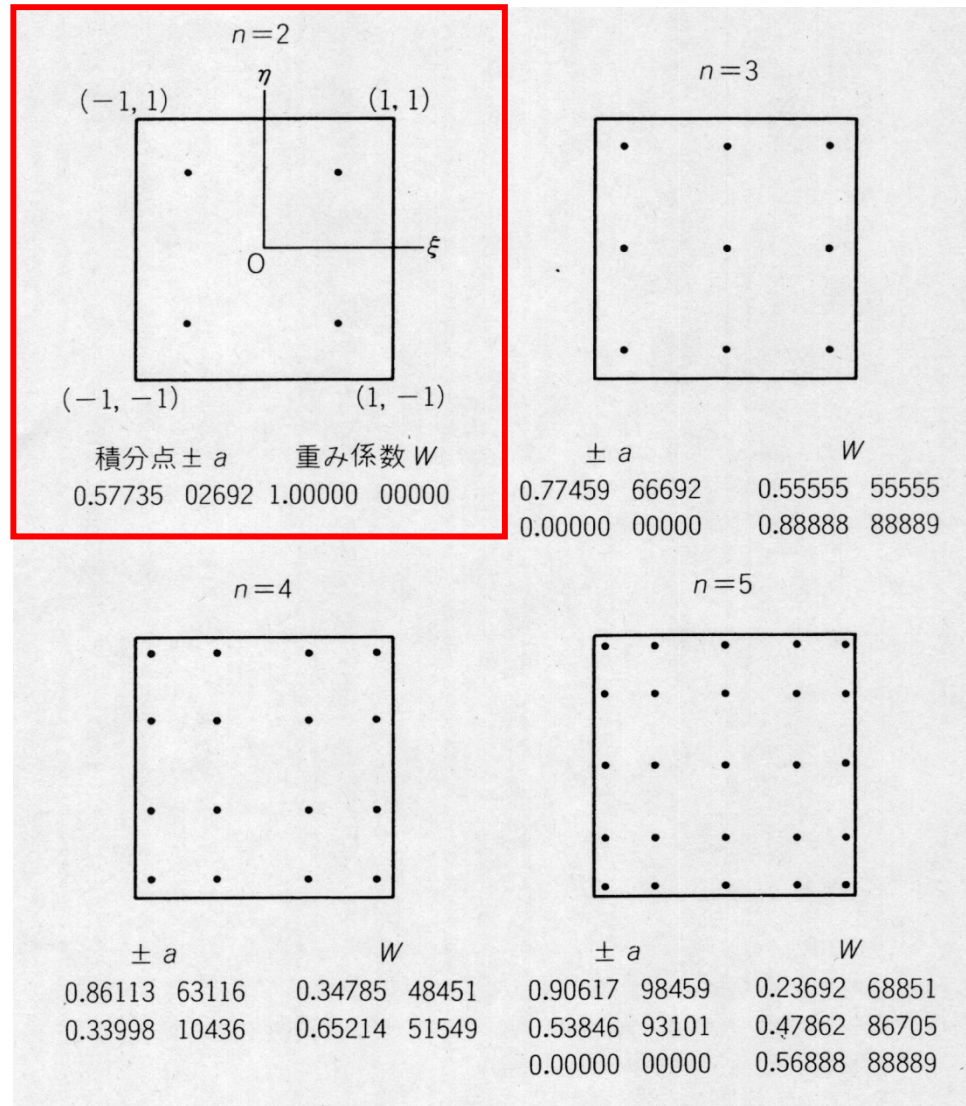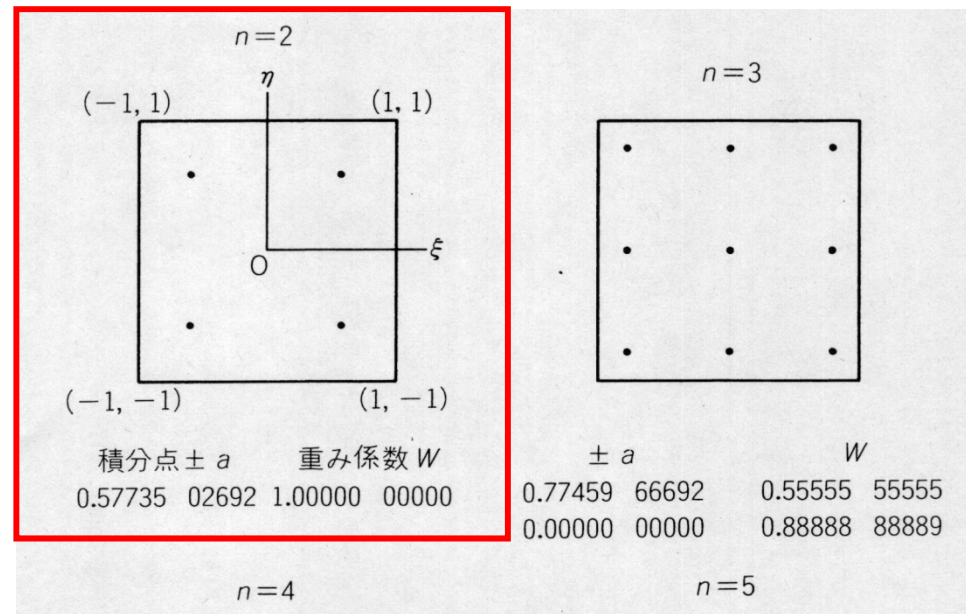# Gaussian Quadrature

This configuration is widely used. In 2D problem, integration is done using values of "f" at 4 quad. points.

Gaussian Quadrature

This configuration is widely used. In 2D problem, integration is done using values of "f" at 4 quad. points.
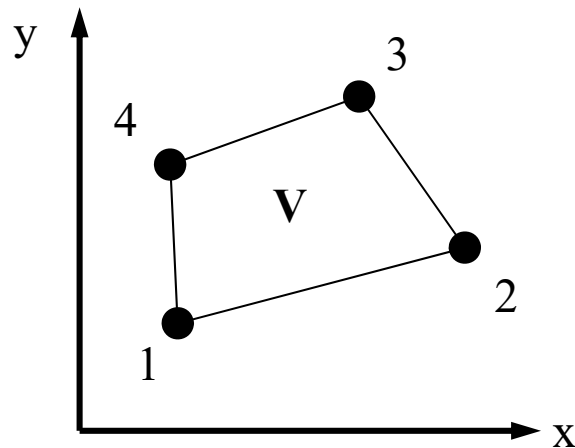

$$ I = \int_{-1}^{+1} \int_{-1}^{+1} f(\xi, \eta) \, d\xi d\eta = \sum_{i=1}^{m} \sum_{j=1}^{n} \left[W_i \cdot W_j \cdot f(\xi_i, \eta_j) \right] $$

$$ = 1.0 \times 1.0 \times f(-0.57735, -0.57735) + 1.0 \times 1.0 \times f(-0.57735, +0.57735) $$

$$ + 1.0 \times 1.0 \times f(+0.57735, +0.57735) + 1.0 \times 1.0 \times f(+0.57735, -0.57735) $$

Homework

- Develop a program and calculate area of the following quadrilateral using Gaussian Quadrature.

$$1: (1.0, \quad 1.0)$$
$$2: (4.0, \quad 2.0)$$
$$3: (3.0, \quad 5.0)$$
$$4: (2.0, \quad 4.0)$$

$$I = \int_V dV$$

Tips

- Calculate Jacobian
- Apply Gaussian Quadrature (n=2)

$$I = \int\limits_{-1}^{+1} \int\limits_{-1}^{+1} f(\xi,\eta)\, d\xi d\eta = \sum_{i=1}^{m} \sum_{j=1}^{n} \left[W_i \cdot W_j \cdot f(\xi_i,\eta_j) \right]$$
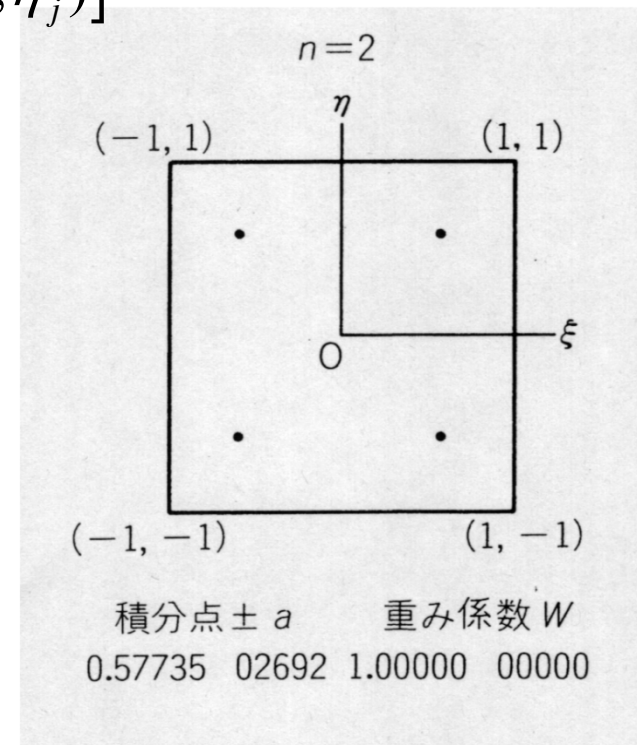
```fortran
implicit REAL*8 (A-H,O-Z)
real*8 W(2)
real*8 POI(2)

W(1)= 1.0d0
W(2)= 1.0d0
POI(1)= -0.5773502692d0
POI(2)= +0.5773502692d0

SUM= 0.d0
do jp= 1, 2
do ip= 1, 2
  FC = F(POI(ip),POI(jp))
  SUM= SUM + W (ip)*W (jp)*FC
enddo
enddo
```

$n=2$

$(-1, 1)$ $(1, 1)$

O ξ

$(-1, -1)$ $(1, -1)$

積分点 ± a 重み係数 W

0.57735 02692   1.00000 00000

Results

1 : (1.0, 1.0)
2 : (4.0, 2.0)
3 : (3.0, 5.0)
4 : (2.0, 4.0)
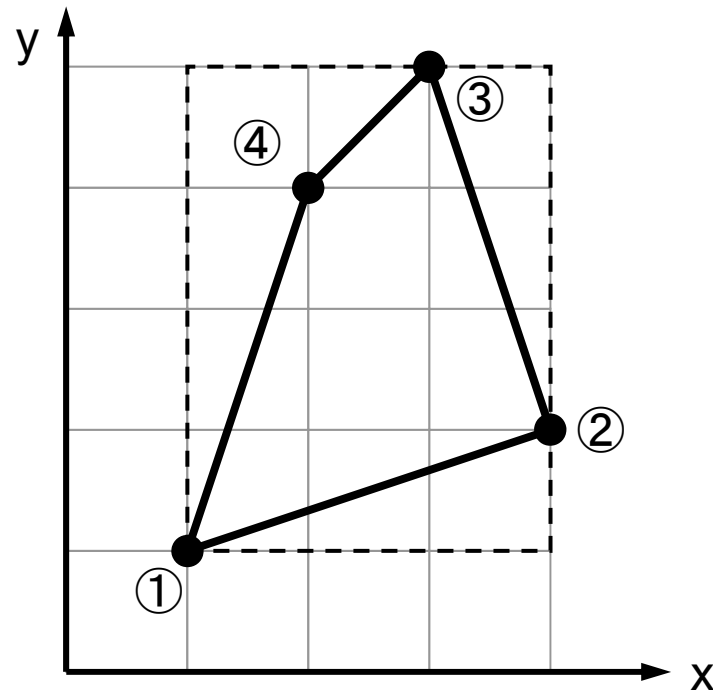
$$3 \times 4 - \frac{1}{2}(3 + 3 + 2 + 4) \times 1$$

$$= 12 - \frac{12}{2} = 6$$

What to do ?

- Final Target:

$$I = \int_V dV = \iint dxdy = \int_{-1}^{+1} \int_{-1}^{+1} \det[J] \, d\xi d\eta$$

- By Definition:

$$I = \int_{-1}^{+1} \int_{-1}^{+1} f(\xi,\eta) \, d\xi d\eta = \sum_{i=1}^{m} \sum_{j=1}^{n} \left[W_i \cdot W_j \cdot f(\xi_i,\eta_j) \right]$$

- Therefore, det $[J]= f$

- Value of det $[J]$ at quad. points should be calculated !

$$\det\left[J(\xi_i,\eta_j) \right]$$

Initialization (1/4)

```
      implicit REAL*8 (A-H,O-Z)

      real*8 X(4), Y(4)
      real*8 W(2), POS(2)
      real*8 SHAPE(2,2,4)
      real*8 PNQ(2,4), PNE(2,4), DETJ(2,2)

!C
!C-- POINT data
      X(1)= 1.0
      Y(1)= 1.0

      X(2)= 4.0
      Y(2)= 2.0

      X(3)= 3.0
      Y(3)= 5.0

      X(4)= 2.0
      Y(4)= 4.0

!C
!C-- Quadrature points & weighting coef.
      W(1)= +1.00000000000d+00
      W(2)= +1.00000000000d+00

      POS(1)= -0.5773502692d+00
      POS(2)= +0.5773502692d+00
```

Node Coord.

1: (1.0, 1.0)
2: (4.0, 2.0)
3: (3.0, 5.0)
4: (2.0, 4.0)

Initialization (1/4)

```
      implicit REAL*8 (A-H,O-Z)

      real*8 X(4), Y(4)
      real*8 W(2), POS(2)
      real*8 SHAPE(2,2,4)
      real*8 PNQ(2,4), PNE(2,4), DETJ(2,2)

!C
!C-- POINT data
      X(1)= 1.0
      Y(1)= 1.0

      X(2)= 4.0
      Y(2)= 2.0

      X(3)= 3.0
      Y(3)= 5.0

      X(4)= 2.0
      Y(4)= 4.0

!C
!C-- Quadrature points & weighting coef.
      W(1)= +1.0000000000d+00
      W(2)= +1.0000000000d+00

      POS(1)= -0.5773502692d+00
      POS(2)= +0.5773502692d+00
```
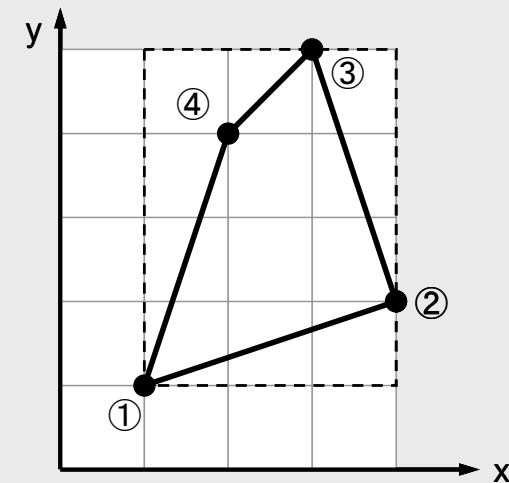
POS : Quad. Points
W : Weighting Factors.

$n=2$

$(-1, 1)$ $(1, 1)$

η

O ξ

$(-1, -1)$ $(1, -1)$

積分点±a 重み係数 W
0.57735 02692 1.00000 00000

Shape Fn., Derivatives at Quad. Points (2/4)

```
!C
!C-- SHAPE functions
     O4th= 0.25d0

     do jp= 1, 2
     do ip= 1, 2
       QP1= 1.d0 + POS(ip)
       QM1= 1.d0 - POS(ip)
       EP1= 1.d0 + POS(jp)
       EM1= 1.d0 - POS(jp)

     SHAPE(ip,jp,1)= O4th * QM1 * EM1
     SHAPE(ip,jp,2)= O4th * QP1 * EM1
     SHAPE(ip,jp,3)= O4th * QP1 * EP1
     SHAPE(ip,jp,4)= O4th * QM1 * EP1

     PNQ(jp,1)= - O4th * EM1
     PNQ(jp,2)= + O4th * EM1
     PNQ(jp,3)= + O4th * EP1
     PNQ(jp,4)= - O4th * EP1

     PNE(ip,1)= - O4th * QM1
     PNE(ip,2)= - O4th * QP1
     PNE(ip,3)= + O4th * QP1
     PNE(ip,4)= + O4th * QM1
     enddo
     enddo
```

$$QP1(i) = (1 + \xi_i), \quad QM1(i) = (1 - \xi_i)$$

$$EP1(j) = (1 + \eta_j), \quad EM1(j) = (1 - \eta_i)$$

Shape Fn., Derivatives at Quad. Points (2/4)

```
!C
!C-- SHAPE functions
     O4th= 0.25d0

    do jp= 1, 2
    do ip= 1, 2
      QP1= 1.d0 + POS(ip)
      QM1= 1.d0 - POS(ip)
      EP1= 1.d0 + POS(jp)
      EM1= 1.d0 - POS(jp)

      SHAPE(ip,jp,1)= O4th * QM1 * EM1
      SHAPE(ip,jp,2)= O4th * QP1 * EM1
      SHAPE(ip,jp,3)= O4th * QP1 * EP1
      SHAPE(ip,jp,4)= O4th * QM1 * EP1

      PNQ(jp,1)= - O4th * EM1
      PNQ(jp,2)= + O4th * EM1
      PNQ(jp,3)= + O4th * EP1
      PNQ(jp,4)= - O4th * EP1

      PNE(ip,1)= - O4th * QM1
      PNE(ip,2)= - O4th * QP1
      PNE(ip,3)= + O4th * QP1
      PNE(ip,4)= + O4th * QM1
    enddo
    enddo
```

$SHAPE$: Values of Shape Fn's.@(ξ_i, η_j)

$$N_1(\xi_i, \eta_j) = \frac{1}{4}(1 - \xi_i)(1 - \eta_j)$$

$$N_2(\xi_i, \eta_j) = \frac{1}{4}(1 + \xi_i)(1 - \eta_j)$$

$$N_3(\xi_i, \eta_j) = \frac{1}{4}(1 + \xi_i)(1 + \eta_j)$$

$$N_4(\xi_i, \eta_j) = \frac{1}{4}(1 - \xi_i)(1 + \eta_j)$$

Shape Fn., Derivatives at Quad. Points (2/4)

```
!C
!C-- SHAPE functions
     O4th= 0.25d0

     do jp= 1, 2
     do ip= 1, 2
       QP1= 1.d0 + POS(ip)
       QM1= 1.d0 - POS(ip)
       EP1= 1.d0 + POS(jp)
       EM1= 1.d0 - POS(jp)

       SHAPE(ip,jp,1)= O4th * QM1 * EM1
       SHAPE(ip,jp,2)= O4th * QP1 * EM1
       SHAPE(ip,jp,3)= O4th * QP1 * EP1
       SHAPE(ip,jp,4)= O4th * QM1 * EP1

       PNQ(jp,1)= - O4th * EM1
       PNQ(jp,2)= + O4th * EM1
       PNQ(jp,3)= + O4th * EP1
       PNQ(jp,4)= - O4th * EP1

       PNE(ip,1)= - O4th * QM1
       PNE(ip,2)= - O4th * QP1
       PNE(ip,3)= + O4th * QP1
       PNE(ip,4)= + O4th * QM1
     enddo
     enddo
```
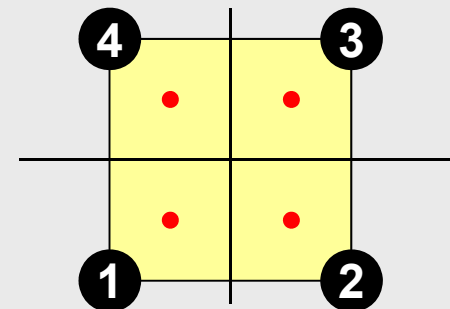
$$PNQ(j,k) = \frac{\partial N_k}{\partial \xi}(\xi = \xi_i, \eta = \eta_j)$$

$$PNE(j,k) = \frac{\partial N_k}{\partial \eta}(\xi = \xi_i, \eta = \eta_j)$$

$$\frac{\partial N_1}{\partial \xi}(\xi_i, \eta_j) = -\frac{1}{4}(1-\eta_j) \qquad \frac{\partial N_1}{\partial \eta}(\xi_i, \eta_j) = -\frac{1}{4}(1-\xi_i)$$

$$\frac{\partial N_2}{\partial \xi}(\xi_i, \eta_j) = +\frac{1}{4}(1-\eta_j) \qquad \frac{\partial N_2}{\partial \eta}(\xi_i, \eta_j) = -\frac{1}{4}(1+\xi_i)$$

$$\frac{\partial N_3}{\partial \xi}(\xi_i, \eta_j) = +\frac{1}{4}(1+\eta_j) \qquad \frac{\partial N_3}{\partial \eta}(\xi_i, \eta_j) = +\frac{1}{4}(1+\xi_i)$$

$$\frac{\partial N_4}{\partial \xi}(\xi_i, \eta_j) = -\frac{1}{4}(1+\eta_j) \qquad \frac{\partial N_4}{\partial \eta}(\xi_i, \eta_j) = +\frac{1}{4}(1-\xi_i)$$

1st order Derivatives at (ξ_i, η_j)

Jacobian at Quad. Points (3/4)

```
!C
!C +-----------------+
!C | JACOBIAN matrix |
!C +-----------------+
!C===
      do jp= 1, 2
      do ip= 1, 2
        dXdQ = PNQ(jp,1)*X(1) + PNQ(jp,2)*X(2) +
     &         PNQ(jp,3)*X(3) + PNQ(jp,4)*X(4)
        dYdQ = PNQ(jp,1)*Y(1) + PNQ(jp,2)*Y(2) +
     &         PNQ(jp,3)*Y(3) + PNQ(jp,4)*Y(4)
        dXdE = PNE(ip,1)*X(1) + PNE(ip,2)*X(2) +
     &         PNE(ip,3)*X(3) + PNE(ip,4)*X(4)
        dYdE = PNE(ip,1)*Y(1) + PNE(ip,2)*Y(2) +
     &         PNE(ip,3)*Y(3) + PNE(ip,4)*Y(4)
        DETJ(ip,jp)= dXdQ*dYdE - dXdE*dYdQ
      enddo
      enddo
!C===
```

$$dXdQ = \frac{\partial x}{\partial \xi} \quad dYdQ = \frac{\partial y}{\partial \xi}$$

$$dXdE = \frac{\partial x}{\partial \eta} \quad dYdE = \frac{\partial y}{\partial \eta}$$

$$DETJ(i,j) = \det\left[J\left(\xi_i, \eta_j\right)\right]$$

$$J_{11} = \frac{\partial x}{\partial \xi} = \frac{\partial}{\partial \xi}\left(\sum_{i=1}^{4} N_i x_i\right) = \sum_{i=1}^{4} \frac{\partial N_i}{\partial \xi} x_i, \quad J_{12} = \frac{\partial y}{\partial \xi} = \frac{\partial}{\partial \xi}\left(\sum_{i=1}^{4} N_i y_i\right) = \sum_{i=1}^{4} \frac{\partial N_i}{\partial \xi} y_i,$$

$$J_{21} = \frac{\partial x}{\partial \eta} = \frac{\partial}{\partial \eta}\left(\sum_{i=1}^{4} N_i x_i\right) = \sum_{i=1}^{4} \frac{\partial N_i}{\partial \eta} x_i, \quad J_{22} = \frac{\partial y}{\partial \eta} = \frac{\partial}{\partial \eta}\left(\sum_{i=1}^{4} N_i y_i\right) = \sum_{i=1}^{4} \frac{\partial N_i}{\partial \eta} y_i$$

Numerical Integration (4/4)

```
!C
!C +------+
!C | AREA |
!C +------+
!C===
      AREA= 0.d0
      do jp= 1, 2
      do ip= 1, 2
        AREA= AREA + dabs(DETJ(ip,jp)) * W(ip) * W(jp)
      enddo
      enddo

!C
!C-- ANALYTICAL SOLUTION
      XA2= X(2) - X(1)
      YA2= Y(2) - Y(1)
      XA3= X(3) - X(1)
      YA3= Y(3) - Y(1)
      XA4= X(4) - X(1)
      YA4= Y(4) - Y(1)

      AREAa= 0.50d0 * (dabs(XA2*YA3-YA2*XA3) +dabs(XA3*YA4-YA3*XA4))
!C===

      write (*,'(a,1pe16.6)') 'Gaussian quadrature', AREA
      write (*,'(a,1pe16.6)') 'analytical sol.    ', AREAa

      stop
      end
```

$$I = \int_{-1}^{+1} \int_{-1}^{+1} f(\xi,\eta)\, d\xi d\eta = \sum_{i=1}^{m} \sum_{j=1}^{n} \left[W_i \cdot W_j \cdot f(\xi_i,\eta_j) \right]$$