1D-FEM in C: Steady State Heat Conduction

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RIKEN R-CCS

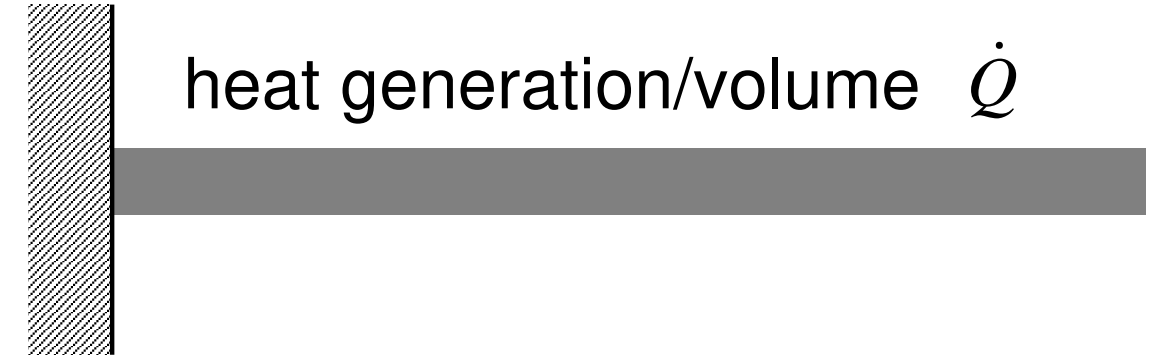
- 1D-code for Steady State Heat Conduction Problems by Galerkin FEM
- Sparse Linear Solver
  - Conjugate Gradient Method
  - Preconditioning
- Storage of Sparse Matrices

- Program

Keywords

- 1D Steady State Heat Conduction Problems
- Galerkin Method
- Linear Element
- Preconditioned Conjugate Gradient Method

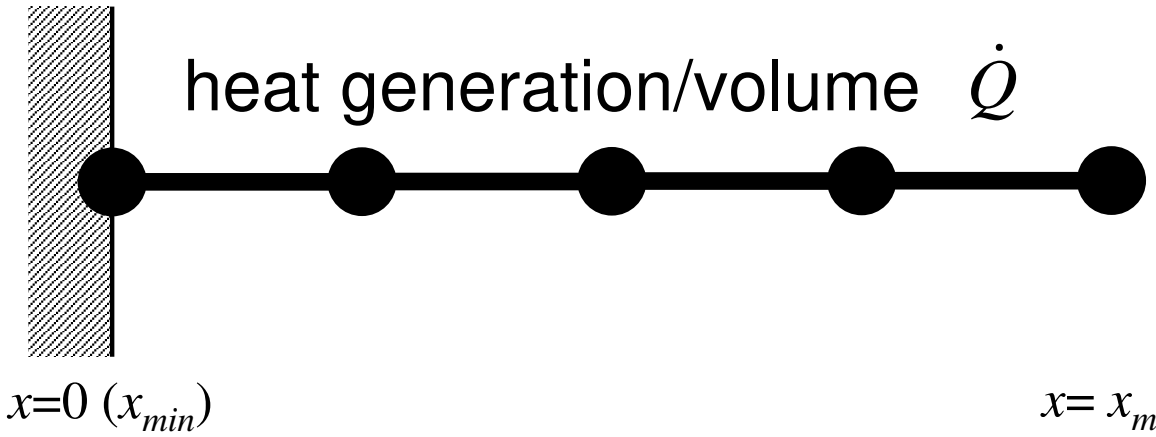
1D Steady State Heat Conduction



$$\frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right)+\dot{Q}=0$$

- Uniform: Sectional Area: A, Thermal Conductivity: λ
- Heat Generation Rate/Volume/Time [$QL^{-3}T^{-1}$] $\dot{Q}$
- Boundary Conditions
  - x=0 : T= 0 (Fixed Temperature)
  - x=x_{max} : $\dfrac{\partial T}{\partial x}=0$ (Insulated)

1D Steady State Heat Conduction



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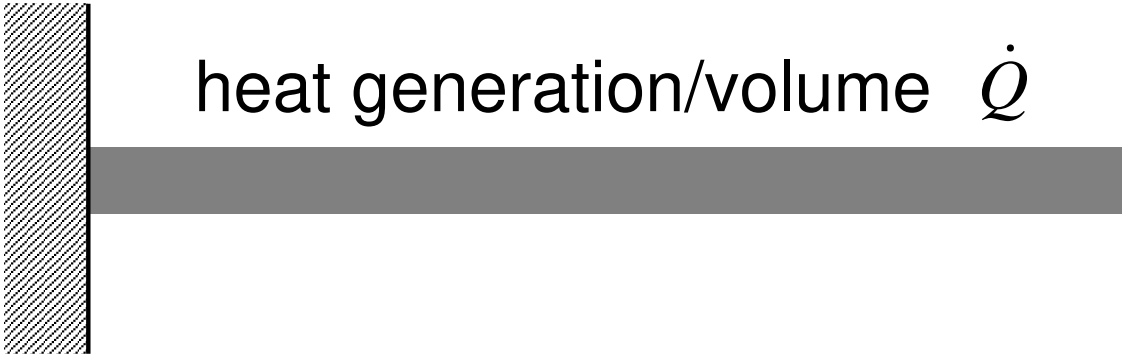
Analytical Solution



$$\frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right)+\dot{Q}=0$$

x=0 (x_{min})

$$T=0 \,@\, x=0$$

x= x_{max}

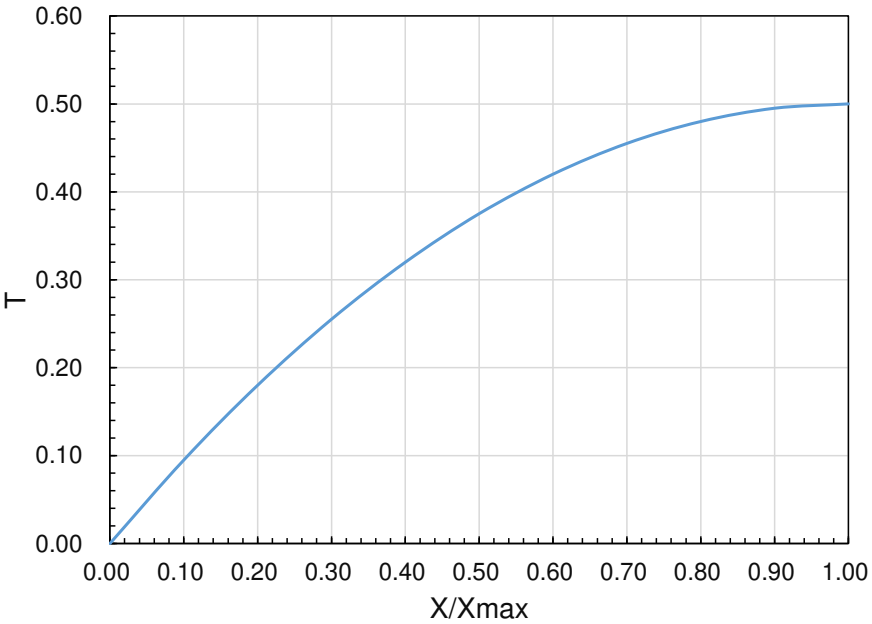
$$\frac{\partial T}{\partial x}=0 \,@\, x=x_{\max}$$

$$\lambda T''=-\dot{Q}$$

$$\lambda T'=-\dot{Q}x+C_1 \Rightarrow C_1=\dot{Q}x_{\max}, \quad T'=0 \,@\, x=x_{\max}$$

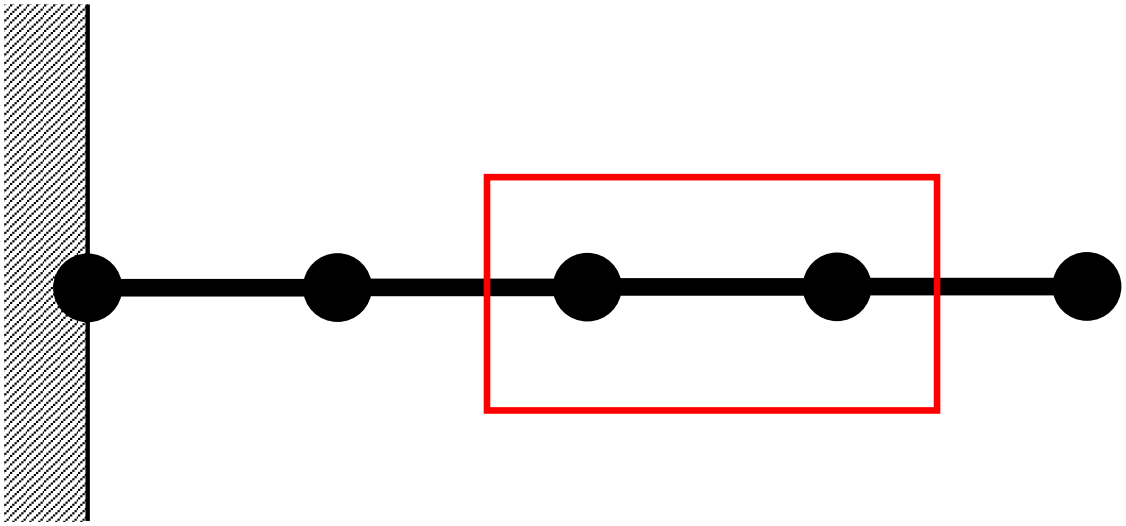
$$\lambda T=-\frac{1}{2}\dot{Q}x^2+C_1x+C_2 \Rightarrow C_2=0, \quad T=0 \,@\, x=0$$

$$\therefore \quad T=-\frac{1}{2\lambda}\dot{Q}x^2+\frac{\dot{Q}x_{\max}}{\lambda}x$$

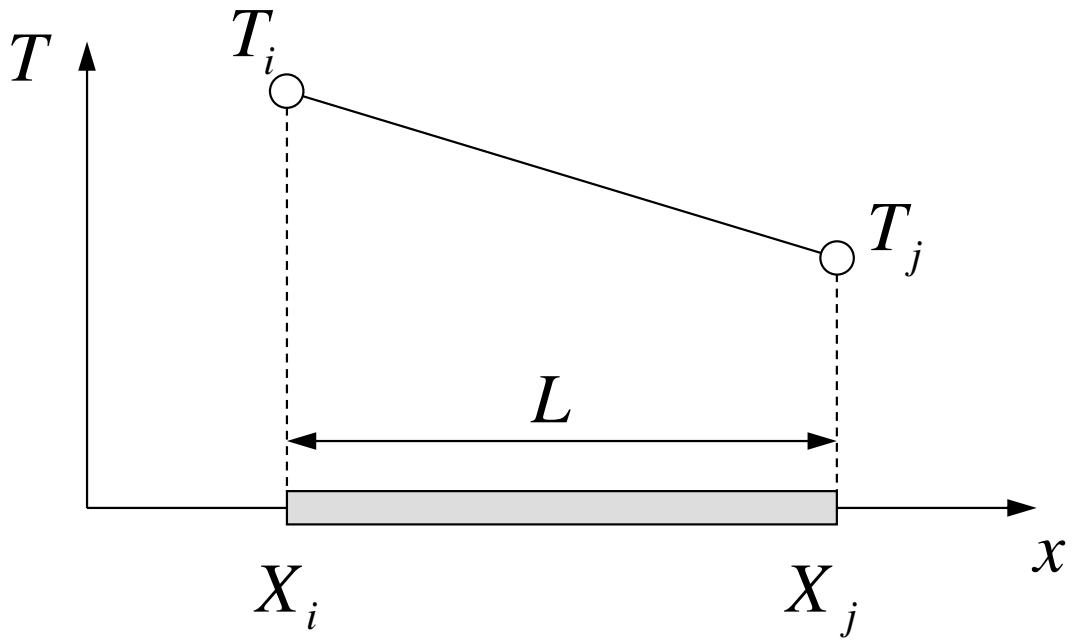


1D Linear Element (1/4)
一次元線形要素

- 1D Linear Element
  - Length= L
    - Node (Vertex)（節点）
    - Element（要素）



  - T_i Temperature at i
  - T_j Temperature at j
  - Temperature T on each element is linear function of x (Piecewise Linear):



$$T = \alpha_1 + \alpha_2 x$$

1D Linear Element (1/4)
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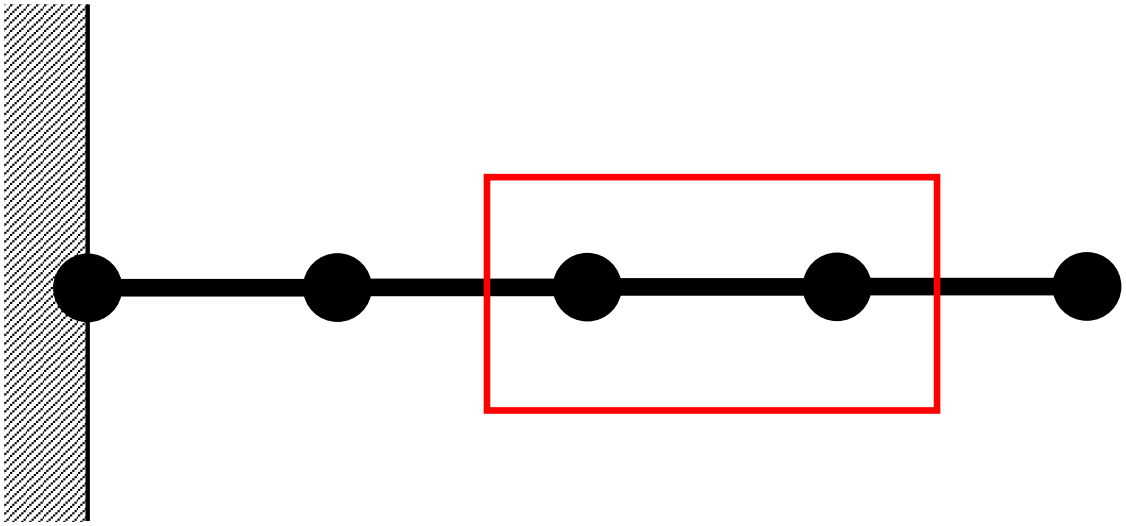
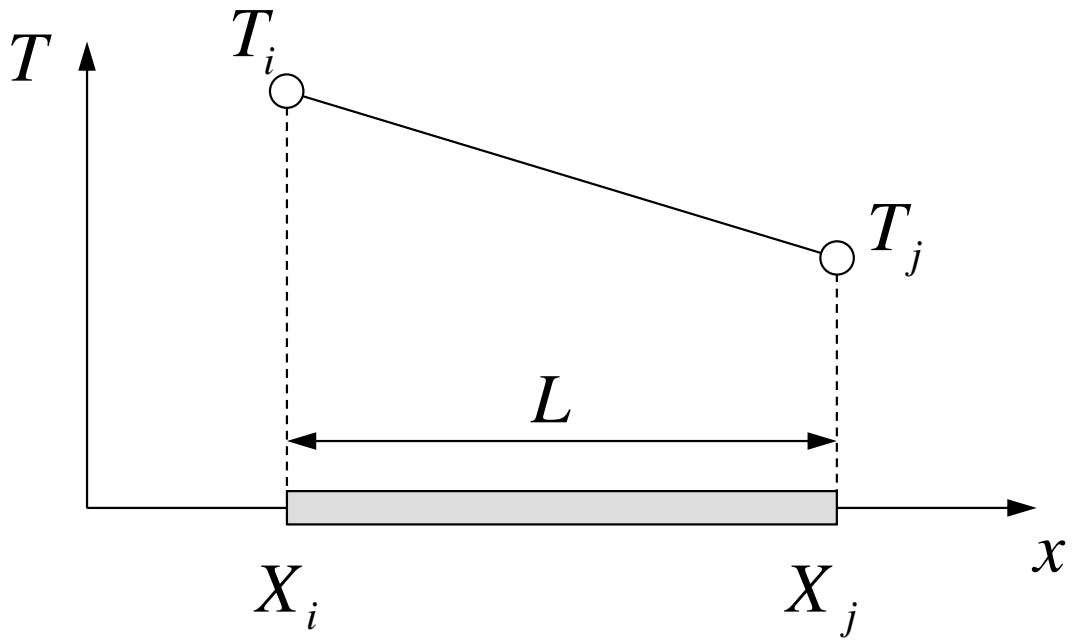
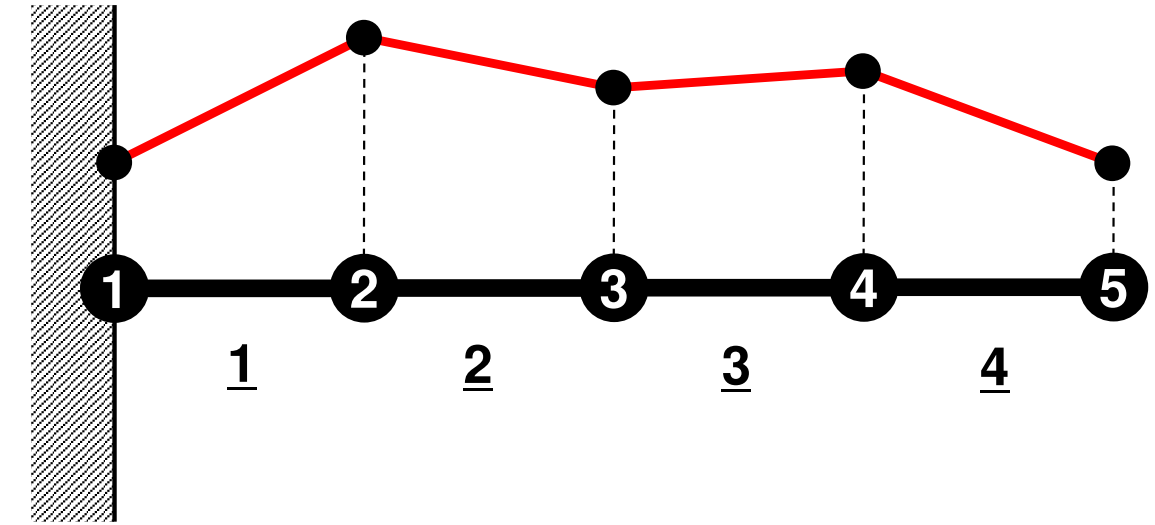
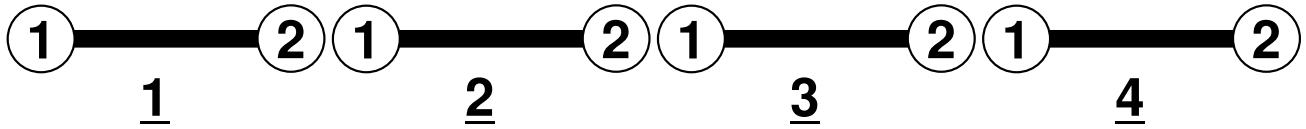




Piecewise Linear



Global Node ID
Element ID



Local Node ID
for each elem.

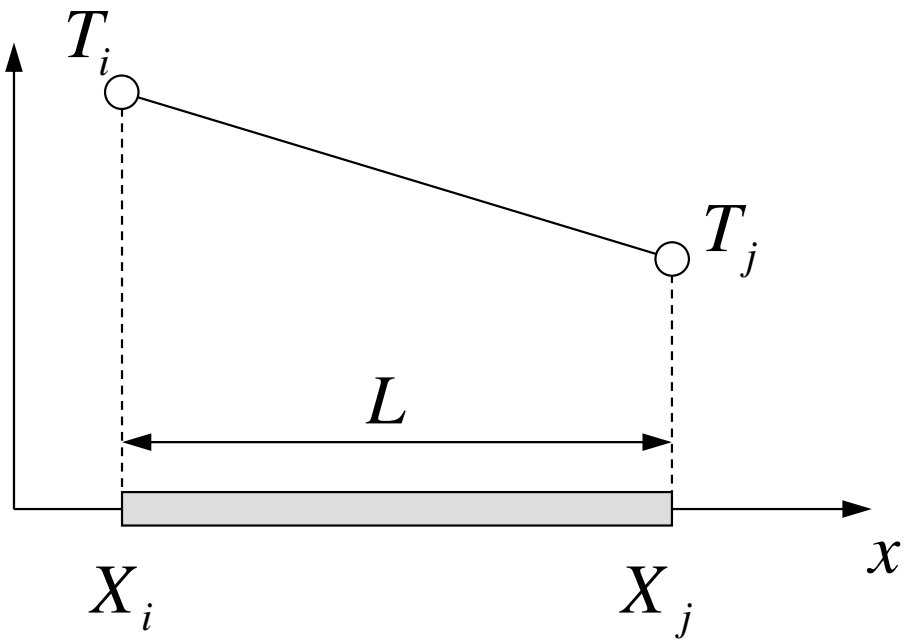
Gradient of temperature is constant in each element (might be discontinuous at each “node”)

1D Linear Elem.: Shape Function (2/4)

- Coef's are calculated based on info. at each node

$$T = T_i \ @\ x = X_i, \quad T = T_j \ @\ x = X_j$$

$$T_i = \alpha_1 + \alpha_2 X_i, \quad T_j = \alpha_1 + \alpha_2 X_j$$



- Coefficients:

$$\alpha_1 = \frac{T_i X_j - T_j X_i}{L}, \alpha_2 = \frac{T_j - T_i}{L}$$

- T can be written as follows, according to T_i and T_j:

$$T = \left(\frac{X_j - x}{L}\right) T_i + \left(\frac{x - X_i}{L}\right) T_j$$

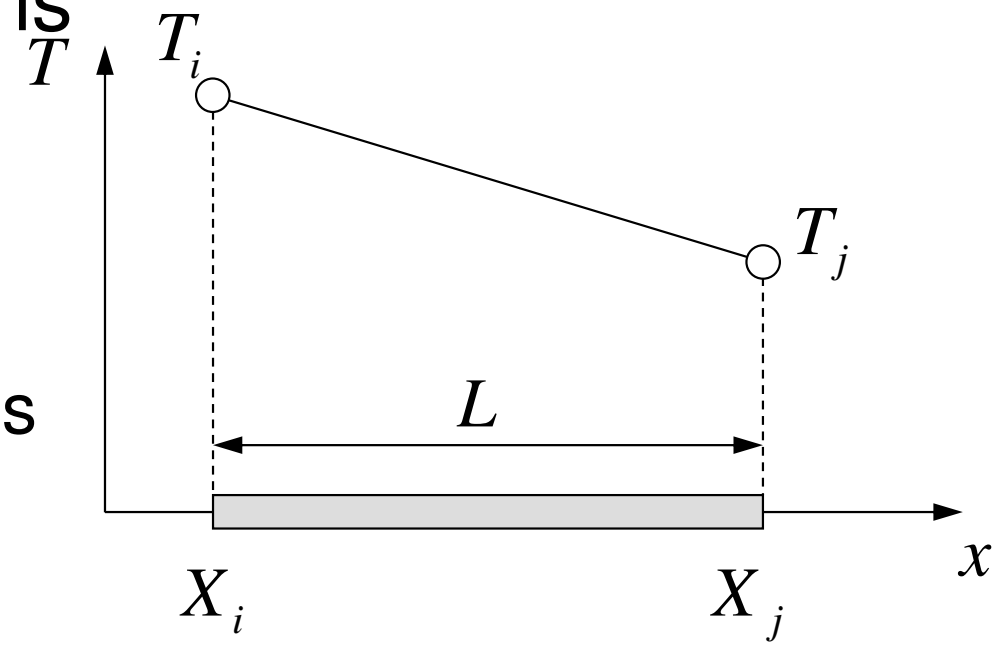
N_i N_j

N_i, N_j
Shape Function（形状関数）or Interpolation Function（内挿関数）, function of x (only)

1D Linear Elem.: Shape Function (3/4)

- Number of Shape Functions = Number of Vertices of Each Element
  - N_i: Function of Position
  - A kind of Test/Trial Functions

$$N_i = \left(\frac{X_j - x}{L} \right), \quad N_j = \left(\frac{x - X_i}{L} \right)$$



- Linear combination of shape functions provides temperature "in" each element
  - Coef's (unknows): Temperature at each node

$$T = N_i T_i + N_j T_j \longleftrightarrow$$

$$T_M = \sum_{i=1}^{M} a_i \Psi_i$$

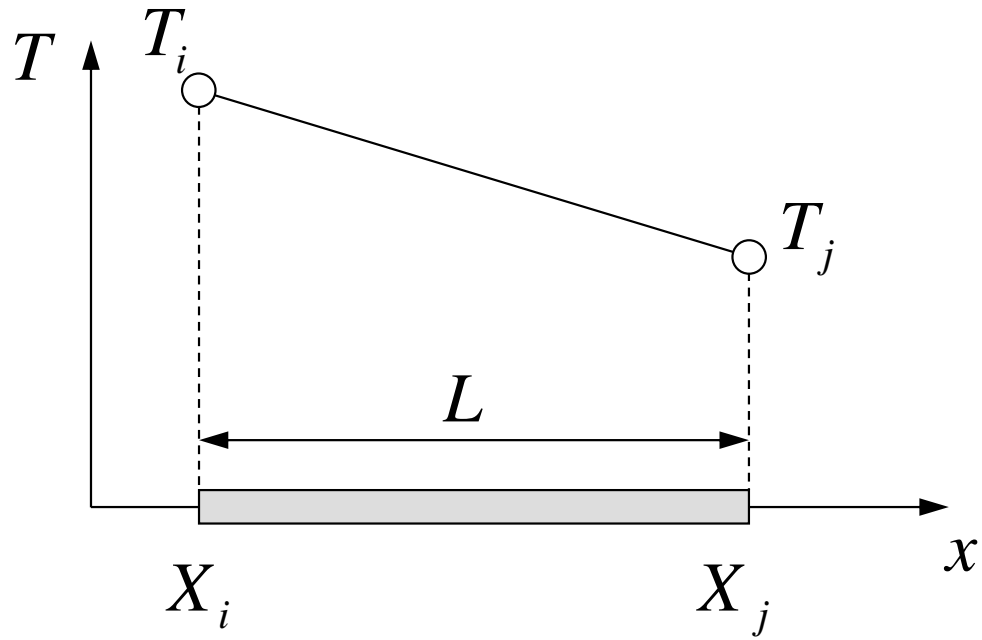
Ψ_i Trial/Test Function (known function of position, defined in domain and at boundary. "Basis" in linear algebra.

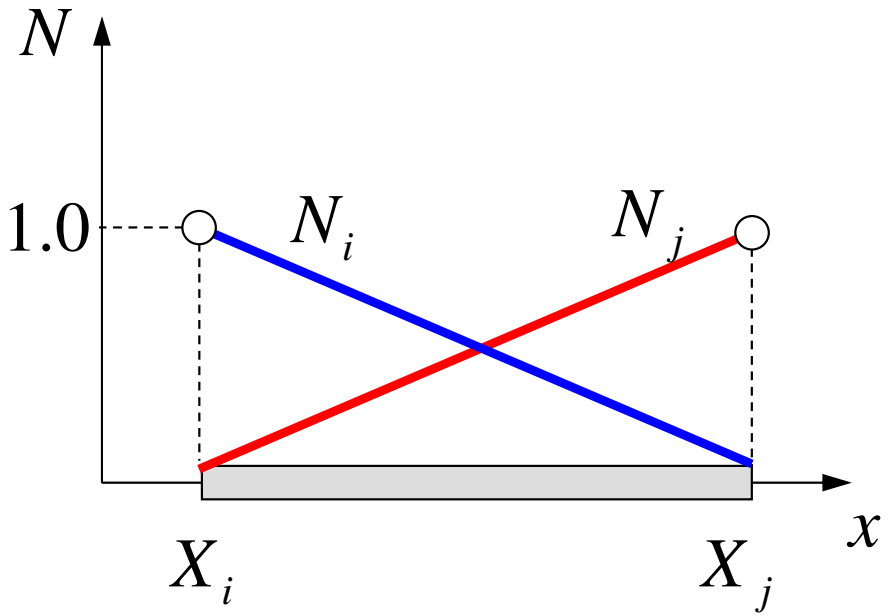
a_i Coefficients (unknown)

1D Linear Elem.: Shape Function (4/4)

- Value of N_i
  - =1 at one of the nodes in element
  - =0 on other nodes

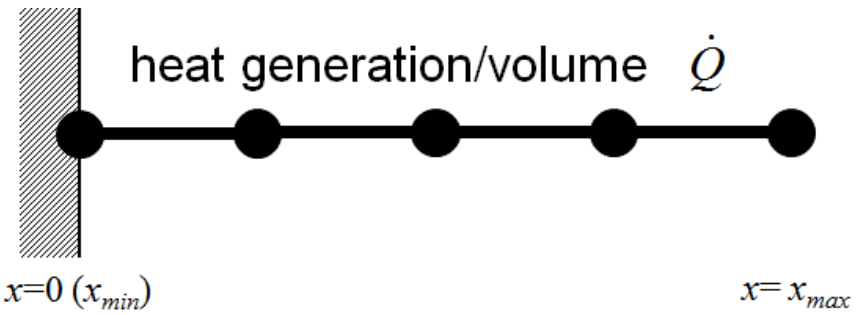
$$N_i = \left(\frac{X_j - x}{L} \right), \quad N_j = \left(\frac{x - X_i}{L} \right)$$





Galerkin Method (1/4)

- Governing Equation for 1D Steady State Heat Conduction Problems (Uniform λ):



$$\lambda\left(\frac{d^2T}{dx^2}\right)+\dot{Q}=0$$

$T=[N]\{\phi\}$ Distribution of temperature in each element (matrix form), ϕ: Temperature at each node

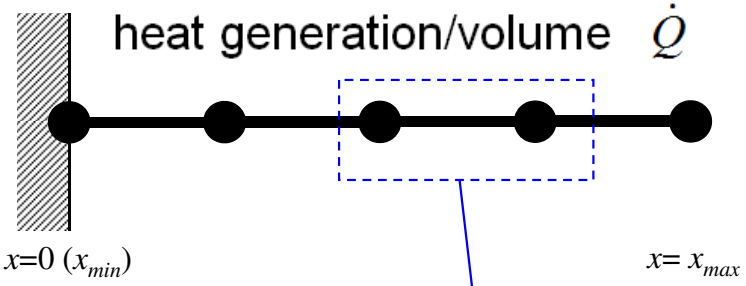
- Following integral equation is obtained at each element by Galerkin method, where $[N]$'s are also weighting functions:

$$\int_V [N]^T\left\{\lambda\left(\frac{d^2T}{dx^2}\right)+\dot{Q}\right\}dV=0$$

Galerkin Method (2/4)

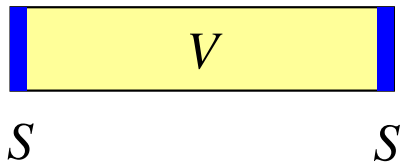
- Green's Theorem (1D)

$$\int_V A\left(\frac{d^2B}{dx^2}\right)dV = \int_S A\frac{dB}{dx}\,dS - \int_V \left(\frac{dA}{dx}\frac{dB}{dx}\right)dV$$



- Apply this to the 1st part of eqn with 2nd-order diff.:

$$\int_V \lambda[N]^T\left(\frac{d^2T}{dx^2}\right)dV = -\int_V \lambda\left(\frac{d[N]^T}{dx}\frac{dT}{dx}\right)dV + \int_S \lambda[N]^T\frac{dT}{dx}\,dS$$



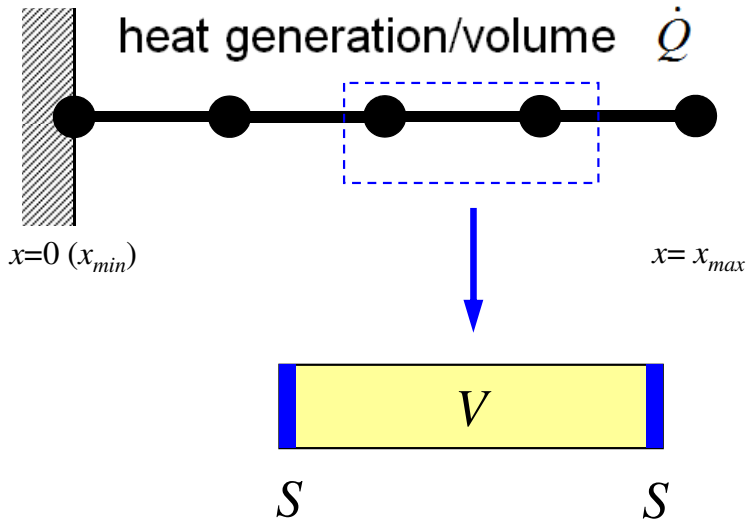
- Consider the following terms:

$$T = [N]\{\phi\}, \quad \frac{dT}{dx} = \frac{d[N]}{dx}\{\phi\} \qquad \bar{q} = -\lambda\frac{dT}{dx}$$

: Heat flux at element surface[$QL^{-2}T^{-1}$]

Galerkin Method (3/4)

- Finally, following eqn is obtained by considering heat generation term $\dot{Q}$:



$$-\int_V \lambda\left(\frac{d[N]^T}{dx}\frac{d[N]}{dx}\right)dV\cdot\{\phi\}$$

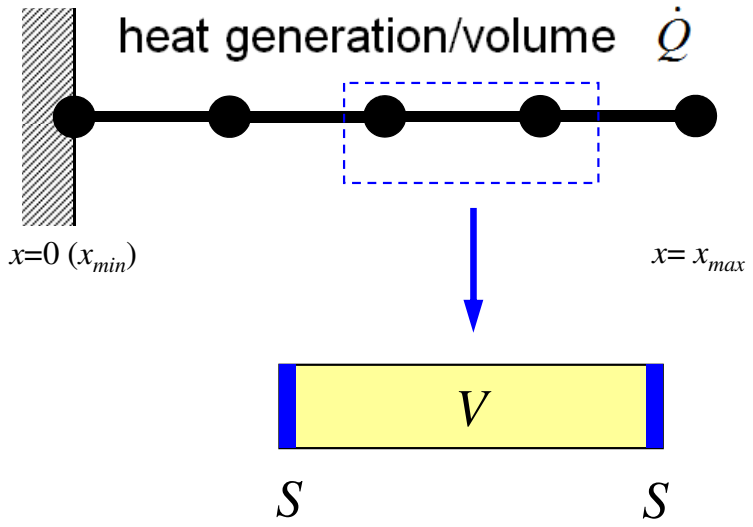
$$-\int_S \bar{q}[N]^T dS+\int_V Q[N]^T dV=0$$

- This is called “weak form（弱形式）”. Original PDE consists of terms with 2nd-order diff., but this “weak form” only includes 1st-order diff by Green’s theorem.
  - Requirements for shape functions are “weaker” in “weak form”. Linear functions can describe effects of 2nd-order differentiation.

Galerkin Method (4/4)

$$-\int_V \lambda\left(\frac{d[N]^T}{dx}\frac{d[N]}{dx}\right)dV\cdot\{\phi\}$$

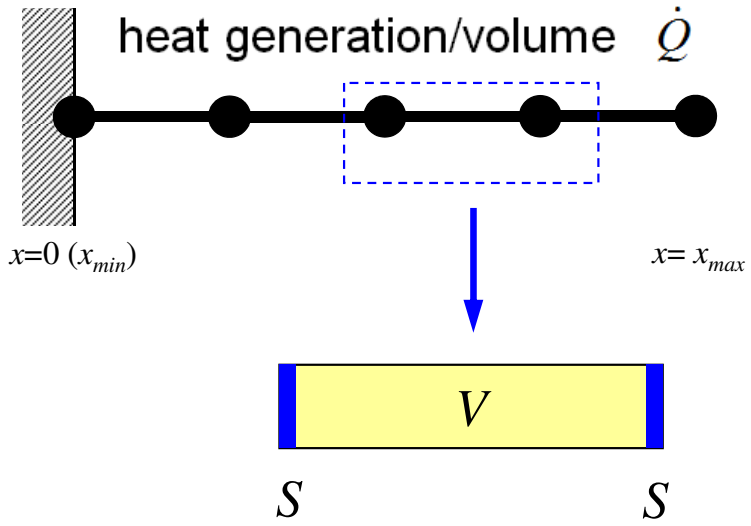
$$-\int_S \bar{q}[N]^T dS+\int_V \dot{Q}[N]^T dV=0$$



- These terms coincide at element boundaries and disappear. Finally, only terms on the domain boundaries remain.

Weak Form and Boundary Conditions

- Value of dependent variable is defined (Dirichlet)
  - Weighting Function = 0
  - Principal B.C. (Boundary Condition)（第一種境界条件）
  - Essential B.C.（基本境界条件）



- Derivatives of Unknowns (Neumann)
  - Naturally satisfied in weak form
  - Secondary B.C.（第二種境界条件）
  - Natural B.C（自然境界条件）

$$-\int_V \lambda\left(\frac{d[N]^T}{dx}\frac{d[N]}{dx}\right)dV\cdot\{\phi\}$$

- (Robin)
  - Linear Combination of Dirichlet & Neumann
  - Third Kind B.C.（第三種境界条件）
  - Electromagnetics

$$-\int_S \bar{q}[N]^T dS+\int_V \dot{Q}[N]^T dV=0$$

where $\bar{q}=-\lambda\frac{dT}{dx}$

Weak Form with B.C.: on each elem.

$$[k]^{(e)}\{\phi\}^{(e)} = \{f\}^{(e)}$$

$$[k]^{(e)} = \int_V \lambda \left(\frac{d[N]^T}{dx} \frac{d[N]}{dx} \right) dV$$

$$[f]^{(e)} = \int_V \dot{Q}[N]^T dV - \int_S \overline{q}[N]^T dS$$

Integration over Each Element: $[k]$

$$N_i = \left(\frac{X_j - x}{L}\right), \quad N_j = \left(\frac{x - X_i}{L}\right)$$

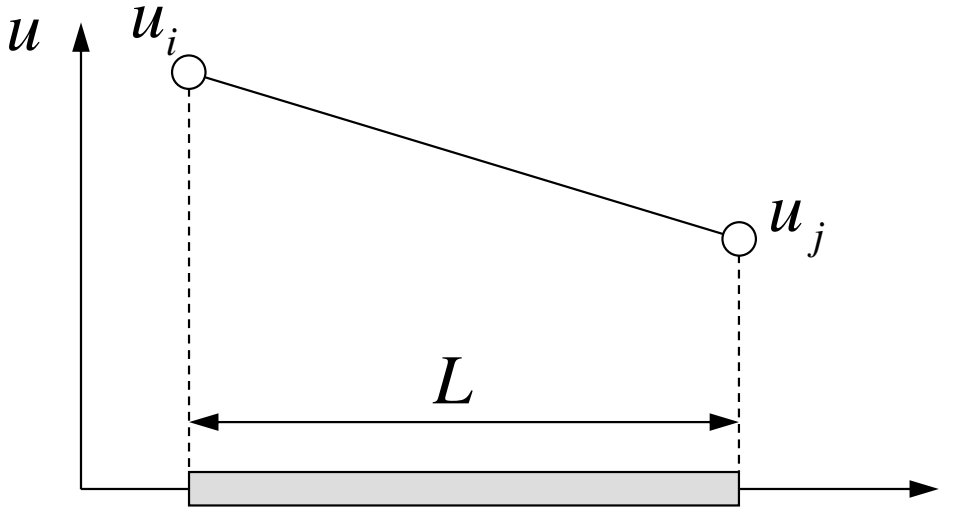
$$\frac{dN_i}{dx} = \left(\frac{-1}{L}\right), \quad \frac{dN_j}{dx} = \left(\frac{1}{L}\right)$$

$$\int_V \lambda \left(\frac{d[N]^T}{dx}\frac{d[N]}{dx}\right) dV$$

$$= \lambda \int_0^L \begin{bmatrix} -1/L \\ 1/L \end{bmatrix} [-1/L, 1/L]\, A\, dx$$

2x1 matrix 1x2 matrix

$$= \frac{\lambda A}{L^2} \int_0^L \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} dx = \frac{\lambda A}{L} \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix}$$



$X_i \Rightarrow 0$ $X_j \Rightarrow L$

$$N_i = \left(1 - \frac{x}{L}\right), \quad N_j = \left(\frac{x}{L}\right)$$

A: Sectional Area

L: Length

Integration over Each Element: $\{f\}$ (1/2)

$$N_i = \left(\frac{X_j - x}{L} \right), \quad N_j = \left(\frac{x - X_i}{L} \right) \qquad \frac{dN_i}{dx} = \left(\frac{-1}{L} \right), \quad \frac{dN_j}{dx} = \left(\frac{1}{L} \right)$$

$$N_i = \left(1 - \frac{x}{L} \right), \quad N_j = \left(\frac{x}{L} \right)$$

$$\int_V \dot{Q}[N]^T dV = \dot{Q}A \int_0^L \begin{bmatrix} 1 - x/L \\ x/L \end{bmatrix} dx = \frac{\dot{Q}AL}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Heat Generation (Volume)

A: Sectional Area

L: Length

Integration over Each Element: $\{f\}$ (2/2)

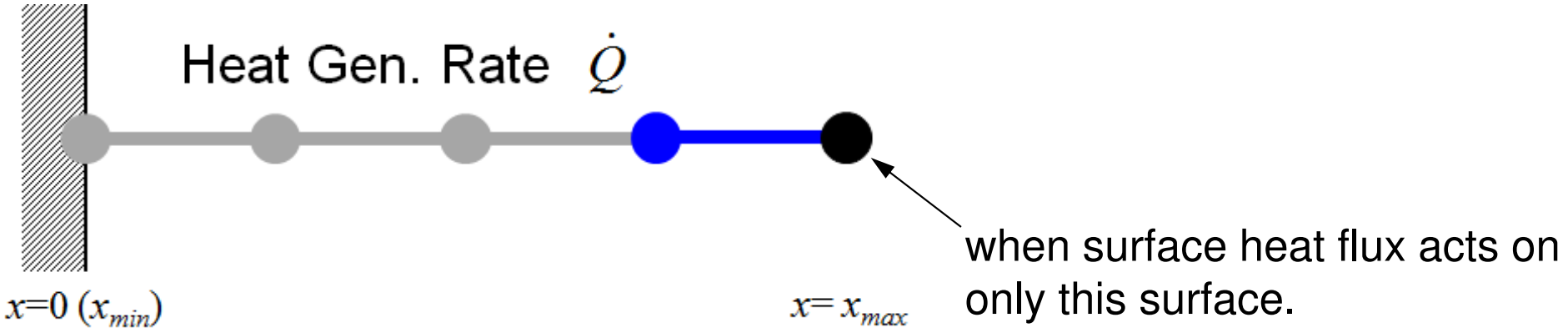
$$N_i = \left(\frac{X_j - x}{L}\right), \quad N_j = \left(\frac{x - X_i}{L}\right) \qquad \frac{dN_i}{dx} = \left(\frac{-1}{L}\right), \quad \frac{dN_j}{dx} = \left(\frac{1}{L}\right)$$

$$\int_V \dot{Q}[N]^T dV = \dot{Q}A\int_0^L \begin{bmatrix} 1 - x/L \\ x/L \end{bmatrix} dx = \frac{\dot{Q}AL}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Heat Generation (Volume)

$$\int_S \bar{q}[N]^T dS = \bar{q}A\big|_{x=L} = \bar{q}A\begin{Bmatrix} 0 \\ 1 \end{Bmatrix}, \quad \bar{q} = -\lambda\frac{dT}{dx}$$

Surface Heat Flux



Global Equations

- Accumulate Element Equations:

$$[k]^{(e)}\{\phi\}^{(e)} = \{f\}^{(e)}$$ Element Matrix, Element Equations

$$[K]\cdot\{\Phi\} = \{F\}$$ Global Matrix, Global Equations

$$[K] = \sum [k], \quad \{F\} = \sum \{f\}$$

$$\{\Phi\}: \textit{global vector of } \{\phi\}$$

This is the final linear equations (global equations) to be solved.

Your PC

Download the Files

(download http://nkl.cc.u-tokyo.ac.jp/files/fem-c.tar)

Copy to ¥Cygwin¥home¥YourName Directory on Windows

1D Code for Steady-State Heat Conduction Problems

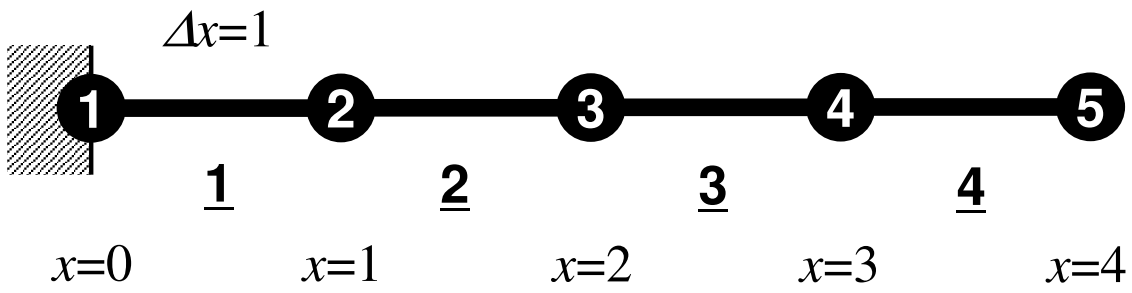
```
>$ cd
>$ tar xvf fem-c.tar
>$ cd fem-c/1d
```

Compile & GO !

```
>$ cd
>$ cd fem-c/1d
>$ cc -O 1d.c (-lm)
>$ ./a.exe (or ./a.out)
```

Control Data `input.dat`

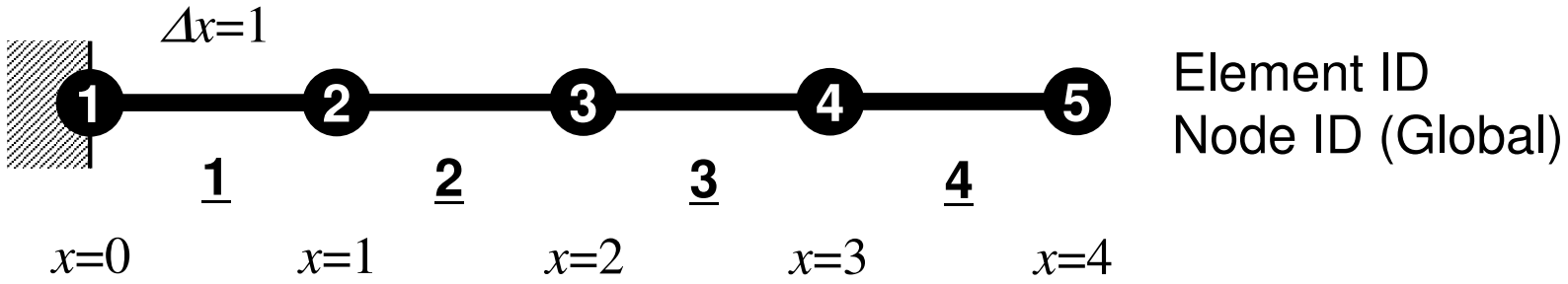
`4`	**NE** (Number of Elements)
`1.0  1.0  1.0  1.0`	**Δx** (Length of Each Elem.: **L**) , **Q**, **A**, λ
`100`	Number of MAX. Iterations for CG Solver
`1.e-8`	Convergence Criteria for CG Solver



Element ID
Node ID (Global)

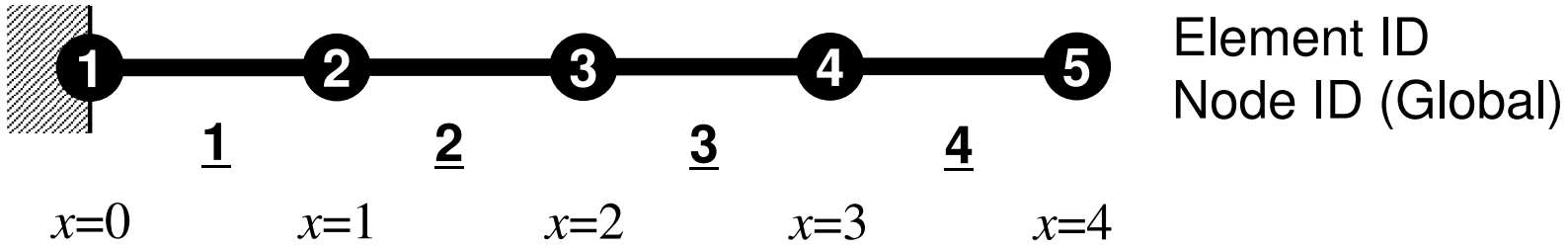
Results

```
>$ ./a.exe (or ./a.out)

  4 iters, RESID=    4.154074e-17

### TEMPERATURE
        1    0.000000E+00    0.000000E+00
        2    3.500000E+00    3.500000E+00
        3    6.000000E+00    6.000000E+00
        4    7.500000E+00    7.500000E+00
        5    8.000000E+00    8.000000E+00

             Computational     Analytical
```



Element Eqn's/Accumulation (1/3)

- 4 elements, 5 nodes



- $[k]$ and $\{f\}$ of Element-1:

$$[k]^{(1)} = \frac{\lambda A}{L}\begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} \qquad \{f\}^{(1)} = \frac{\dot{Q}AL}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

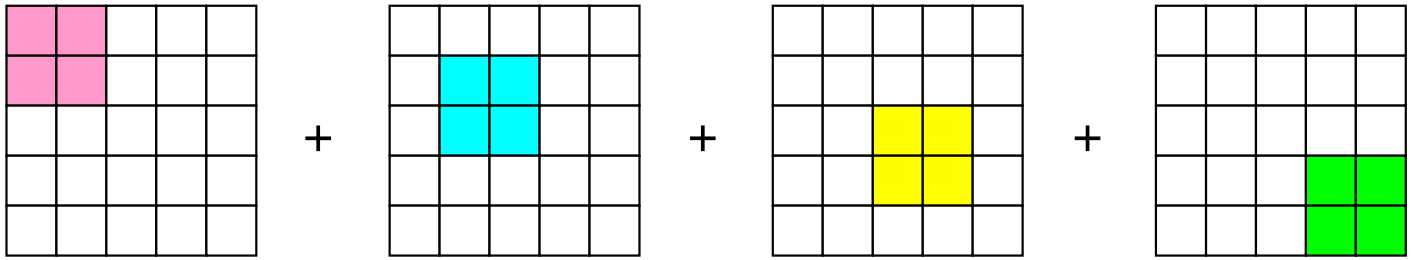
- As for Element-4:

$$[k]^{(4)} = \frac{\lambda A}{L}\begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} \qquad \{f\}^{(4)} = \frac{\dot{Q}AL}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

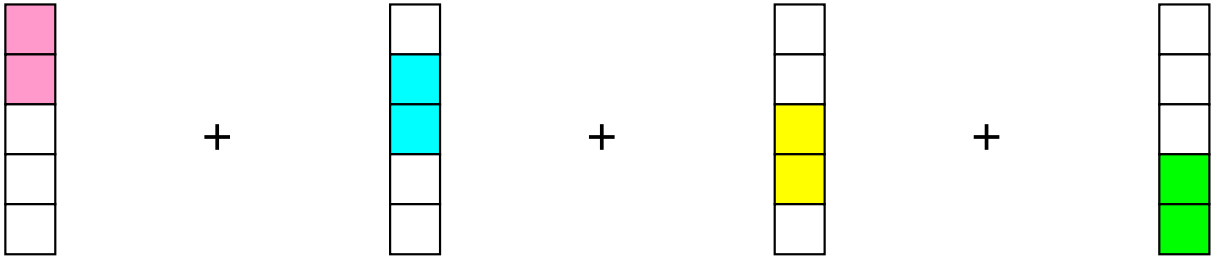
Element Eqn's/Accumulation (2/3)

- Element-by-Element Accumulation:

$$[K] = \sum_{e=1}^{4} [k]^{(e)} =$$



$$\{F\} = \sum_{e=1}^{4} \{f\}^{(e)} =$$



Element Eqn's/Accumulation (3/3)

- Relations to FDM

$$[k]^{(e)} = \frac{\lambda A}{L}\begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix}$$

$$[K] = \sum_{e=1}^{4} [k]^{(e)} = \left[\begin{array}{|c|c|c|c|c|} \hline +1 & -1 & & & \\ \hline -1 & +1 & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline & +1 & -1 & & \\ \hline & -1 & +1 & & \\ \hline & & & & \\ \hline & & & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline & & & & \\ \hline & & +1 & -1 & \\ \hline & & -1 & +1 & \\ \hline & & & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline & & & & \\ \hline & & & +1 & -1 \\ \hline & & & -1 & +1 \\ \hline & & & & \\ \hline \end{array} \right] \times \frac{\lambda A}{L}$$

$$= \begin{array}{|c|c|c|c|c|} \hline +1 & -1 & & & \\ \hline -1 & +2 & -1 & & \\ \hline & -1 & +2 & -1 & \\ \hline & & -1 & +2 & -1 \\ \hline & & & -1 & +1 \\ \hline \end{array} \times \frac{\lambda A}{L}$$

三重対角行列
Tri-Diagonal Matrix

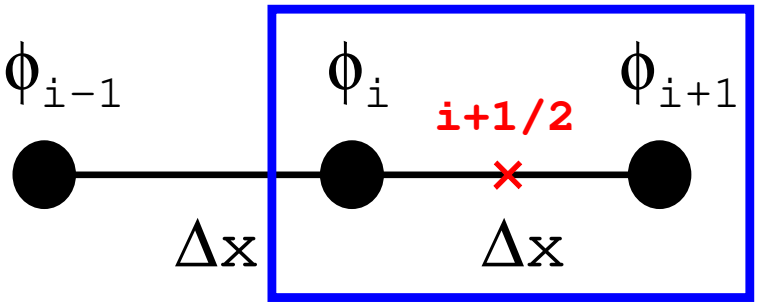
$$-\int_V \left(\frac{d^2T}{dx^2}\right) dV = -\int_V \left(\frac{T_{i+1} - 2T_i + T_{i-1}}{L^2}\right) dV$$

$$= -\left(\frac{T_{i+1} - 2T_i + T_{i-1}}{L^2}\right) \cdot AL = -(T_{i+1} - 2T_i + T_{i-1}) \cdot \frac{A}{L}$$

Something familiar …
FEM: Coefficient Matrices are generally sparse (many ZERO's)

2nd Order Differentiation in FDM

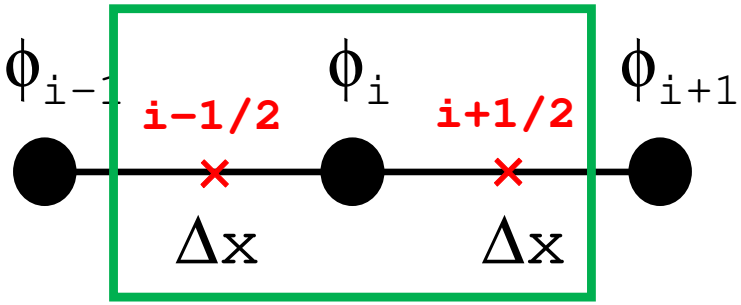
- Approximate Derivative at × (center of i and $i+1$)



$$\left(\frac{d\phi}{dx}\right)_{i+1/2} \approx \frac{\phi_{i+1}-\phi_i}{\Delta x}$$

$\Delta x \to 0$: Real Derivative

- 2nd-Order Diff. at i



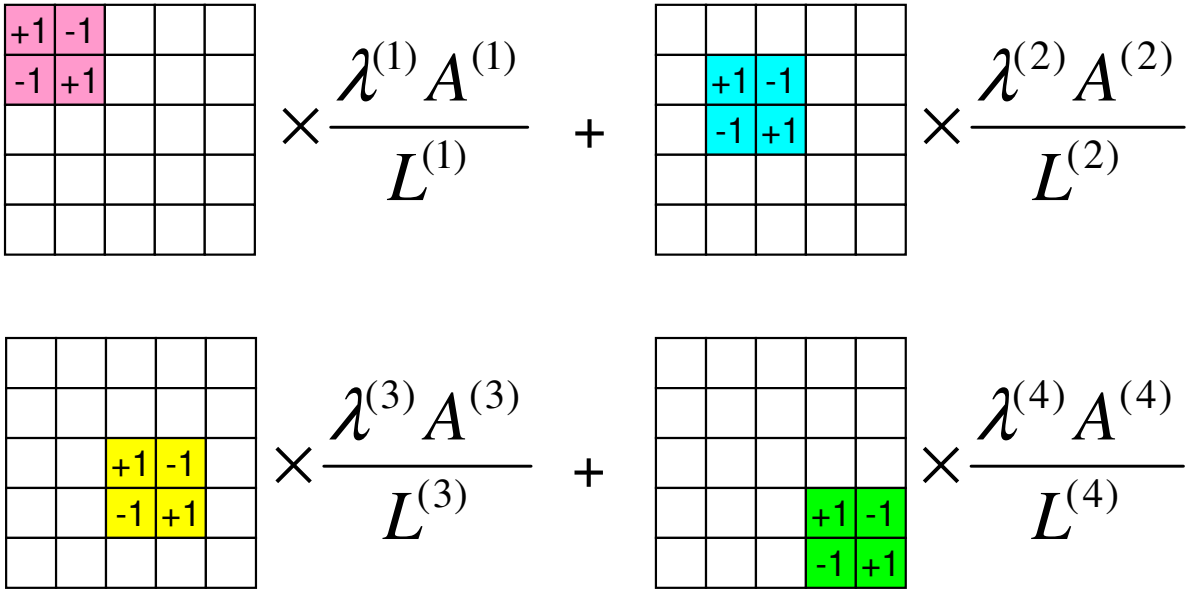
$$\left(\frac{d^2\phi}{dx^2}\right)_i \approx \frac{\left(\frac{d\phi}{dx}\right)_{i+1/2} - \left(\frac{d\phi}{dx}\right)_{i-1/2}}{\Delta x} = \frac{\frac{\phi_{i+1}-\phi_i}{\Delta x} - \frac{\phi_i-\phi_{i-1}}{\Delta x}}{\Delta x} = \frac{\phi_{i+1}-2\phi_i+\phi_{i-1}}{\Delta x^2}$$

Element-by-Element Operation

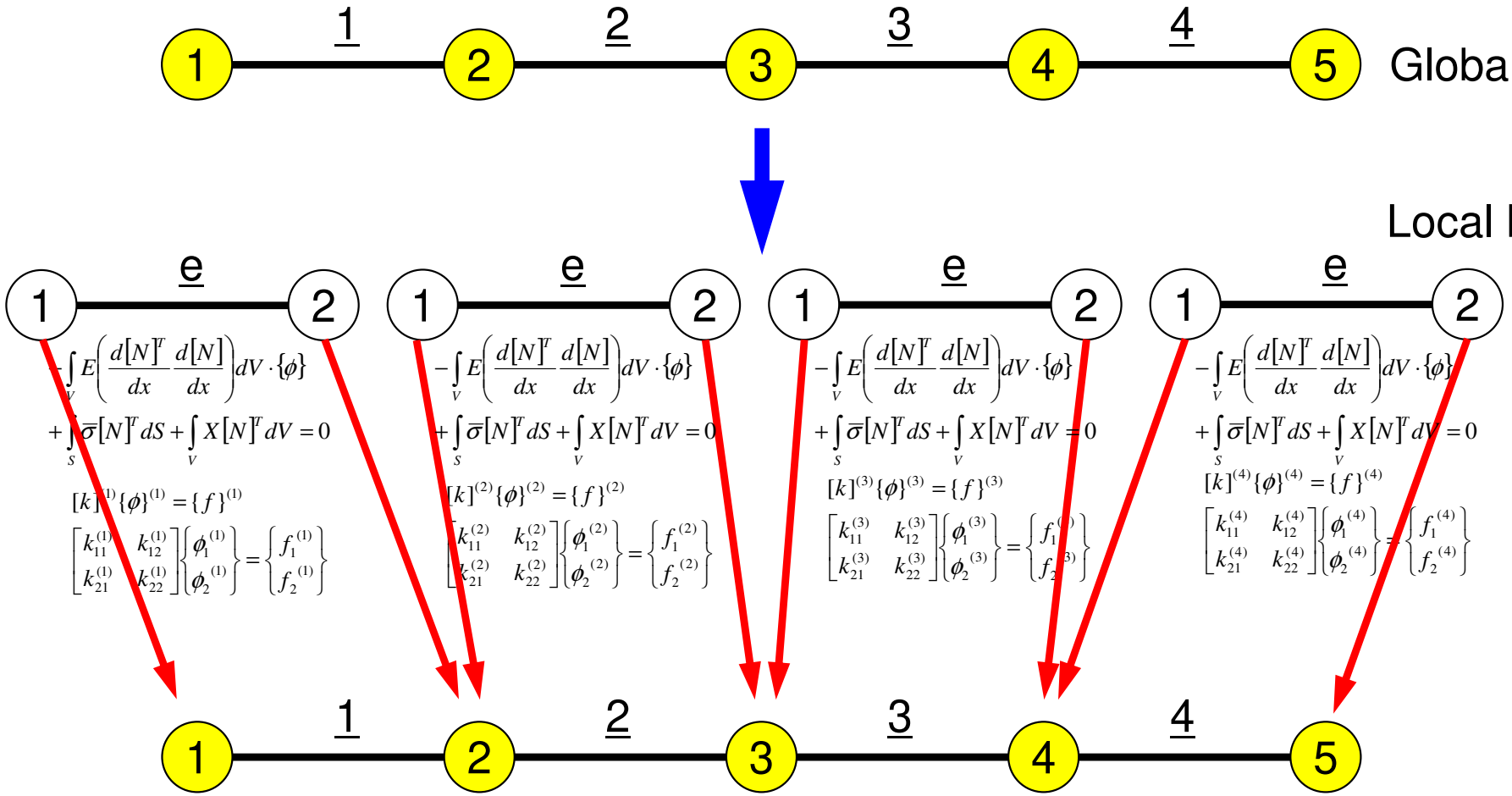
very flexible if each element has different material property, size, etc.

$$[k]^{(e)} = \frac{\lambda^{(e)} A^{(e)}}{L^{(e)}} \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix}$$

$$[K] = \sum_{e=1}^{4} [k^{(e)}] =$$



Element/Global Operations



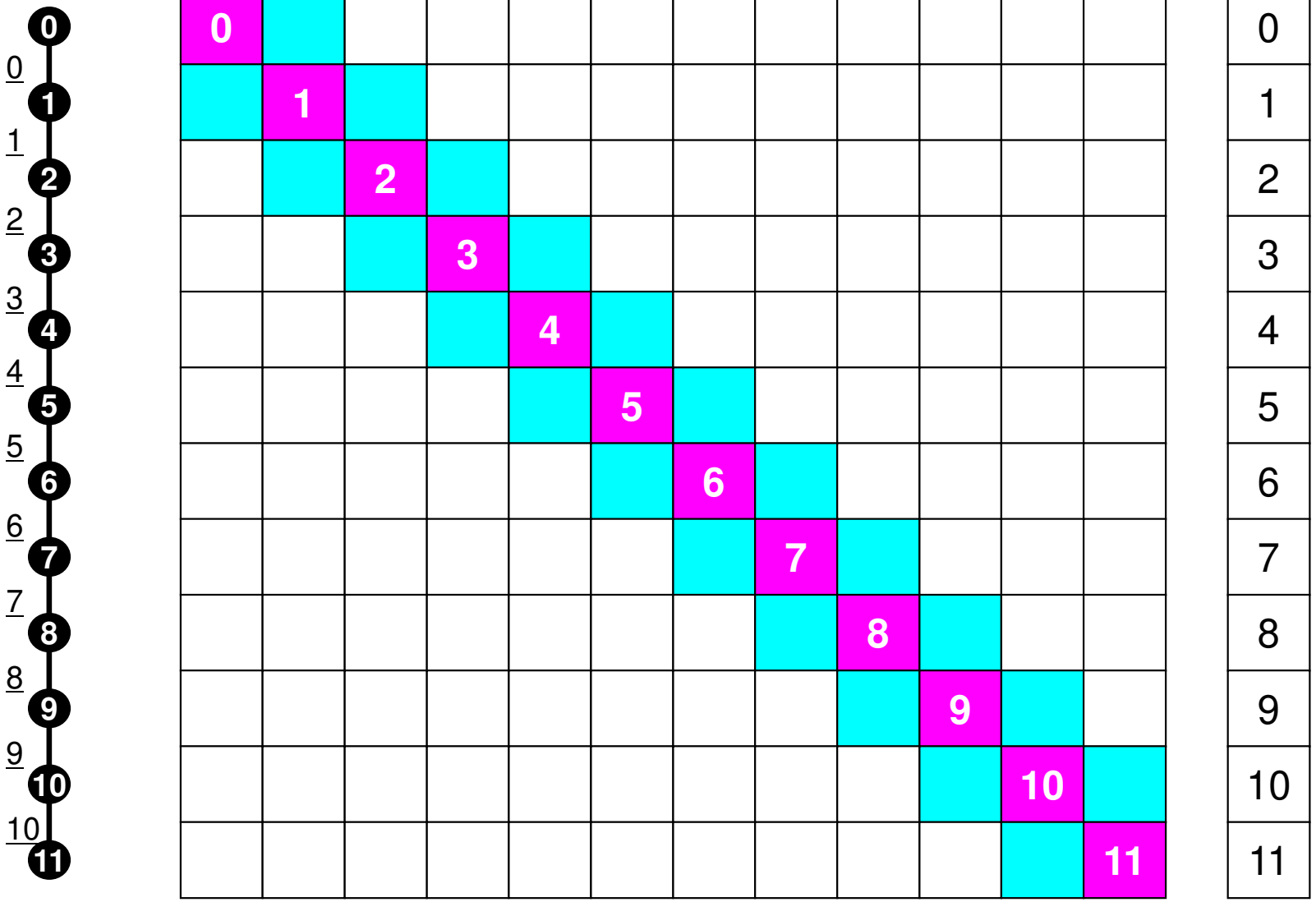
$$[K]\{\Phi\}=\{F\}$$

$$\begin{bmatrix} D_1 & AU_{11} & & & \\ AL_{21} & D_2 & AU_{21} & & \\ & AL_{31} & D_3 & AU_{31} & \\ & & AL_{41} & D_4 & AU_{41} \\ & & & AL_{51} & D_5 \end{bmatrix}\begin{Bmatrix}\Phi_1\\ \Phi_2\\ \Phi_3\\ \Phi_4\\ \Phi_5\end{Bmatrix}=\begin{Bmatrix}B_1\\ B_2\\ B_3\\ B_4\\ B_5\end{Bmatrix}$$

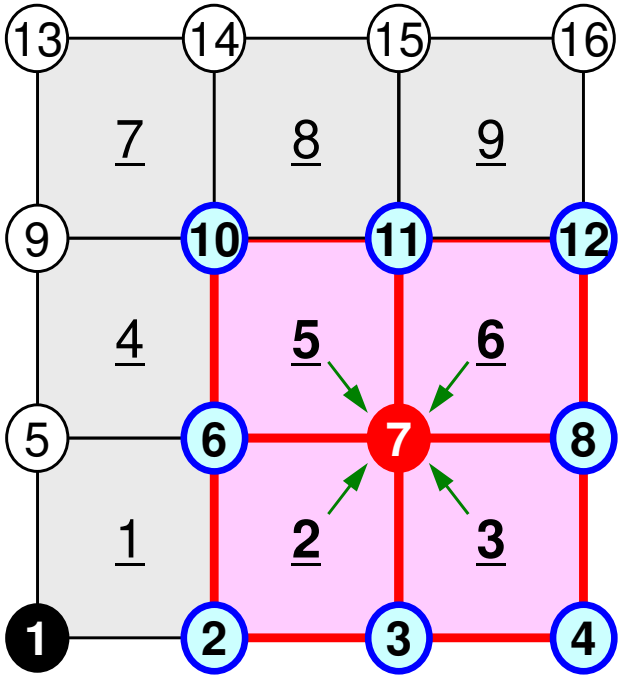
Mapping Information needed, from element-matrix to global-matrix.

1D FEM: 12 nodes/11 elem's

三重対角行列 : Tri-Diagonal Matrix



Accumulation to Global/overall Matrix



$$[K]\{\Phi\}=\{F\}$$

$$\begin{bmatrix}
D & X & & & X & X & & & & & & & & & & \\
X & D & X & & X & X & X & & & & & & & & & \\
 & X & D & X & & X & X & X & & & & & & & & \\
 & & X & D & & & X & X & & & & & & & & \\
X & X & & & D & X & & & X & X & & & & & & \\
X & X & X & & X & D & X & & X & X & X & & & & & \\
 & X & X & X & & X & D & X & & X & X & X & & & & \\
 & & X & X & & & X & D & & & X & X & & & & \\
 & & & & X & X & & & D & X & & & X & X & & \\
 & & & & X & X & X & & X & D & X & & X & X & X & \\
 & & & & & X & X & X & & X & D & X & & X & X & X \\
 & & & & & & X & X & & & X & D & & & X & X \\
 & & & & & & & & X & X & & & D & X & & \\
 & & & & & & & & X & X & X & & X & D & X & \\
 & & & & & & & & & X & X & X & & X & D & X \\
 & & & & & & & & & & X & X & & & X & D
\end{bmatrix}
\begin{Bmatrix}\Phi_1\\ \Phi_2\\ \Phi_3\\ \Phi_4\\ \Phi_5\\ \Phi_6\\ \Phi_7\\ \Phi_8\\ \Phi_9\\ \Phi_{10}\\ \Phi_{11}\\ \Phi_{12}\\ \Phi_{13}\\ \Phi_{14}\\ \Phi_{15}\\ \Phi_{16}\end{Bmatrix}
=
\begin{Bmatrix}F_1\\ F_2\\ F_3\\ F_4\\ F_5\\ F_6\\ F_7\\ F_8\\ F_9\\ F_{10}\\ F_{11}\\ F_{12}\\ F_{13}\\ F_{14}\\ F_{15}\\ F_{16}\end{Bmatrix}$$

- 1D-code for Static Linear-Elastic Problems by Galerkin FEM
- Sparse Linear Solver
  - Conjugate Gradient Method
  - Preconditioning
- Storage of Sparse Matrices

- Program

Large-Scale Linear Equations in Scientific Applications

- Solving large-scale linear equations **Ax=b** is the most important and **expensive** part of various types of scientific computing.
  - for both linear and nonlinear applications
- Various types of methods proposed & developed.
  - for dense and sparse matrices
  - classified into ***direct*** and ***iterative*** methods
- Dense Matrices:密行列: Globally Coupled Problems
  - BEM, Spectral Methods, MO/MD (gas, liquid)
- Sparse Matrices:疎行列: Locally Defined Problems
  - FEM, FDM, DEM, MD (solid), BEM w/FMM

Direct Method
直接法

- Gaussian Elimination/LU Factorization.
  - compute $\mathbf{A}^{-1}$ directly (or equivalent operations)

Good

  - Robust for wide range of applications.
  - Good for both dense and sparse matrices

Bad

  - More expensive than iterative methods (memory, CPU)
    - not scalable

What is Iterative Method ?
反復法

Linear Equations
連立一次方程式

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$\mathbf{A} \qquad \mathbf{x} \qquad \mathbf{b}$$

Initial Solution
初期解

$$\mathbf{x}^{(0)} = \begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \\ \vdots \\ x_n^{(0)} \end{pmatrix}$$

Starting from a initial vector $\mathbf{x}^{(0)}$, iterative method obtains the final converged solutions by iterations

$$\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \cdots$$

Iterative Method
反復法

- Stationary Method
  - Only $\mathbf{x}$ (solution vector) changes during iterations.
  - SOR, Gauss-Seidel, Jacobi
  - Generally slow, impractical

$$\mathbf{Ax} = \mathbf{b} \Rightarrow$$
$$\mathbf{x}^{(k+1)} = \mathbf{Mx}^{(k)} + \mathbf{Nb}$$

- Non-Stationary Method
  - With restriction/optimization conditions
  - Krylov-Subspace
  - CG: Conjugate Gradient
  - BiCGSTAB: Bi-Conjugate Gradient Stabilized
  - GMRES: Generalized Minimal Residual

Iterative Method (cont.)

Good

- Less expensive than direct methods, especially in memory.
- Suitable for parallel and vector computing.

Bad

- Convergence strongly depends on problems, boundary conditions (condition number etc.)
- **Preconditioning is required : Key Technology for Parallel FEM**

Non-Stationary/Krylov Subspace Method (1/2)

非定常法・クリロフ部分空間法

$$\mathbf{Ax} = \mathbf{b} \Rightarrow \mathbf{x} = \mathbf{b} + (\mathbf{I} - \mathbf{A})\mathbf{x}$$

Compute $\mathbf{x_0}, \mathbf{x_1}, \mathbf{x_2}, ..., \mathbf{x_k}$ by the following iterative procedures:

$$\begin{aligned}\mathbf{x}_k &= \mathbf{b} + (\mathbf{I} - \mathbf{A})\mathbf{x}_{k-1} \\ &= (\mathbf{b} - \mathbf{A}\mathbf{x}_{k-1}) + \mathbf{x}_{k-1} \\ &= \mathbf{r}_{k-1} + \mathbf{x}_{k-1}\end{aligned}$$

$where\ \mathbf{r}_k = \mathbf{b} - \mathbf{A}\mathbf{x}_k$: residual

$$\mathbf{x}_k = \mathbf{x}_0 + \sum_{i=0}^{k-1} \mathbf{r}_i$$

$$\begin{aligned}\mathbf{r}_k &= \mathbf{b} - \mathbf{A}\mathbf{x}_k = \mathbf{b} - \mathbf{A}(\mathbf{r}_{k-1} + \mathbf{x}_{k-1}) \\ &= (\mathbf{b} - \mathbf{A}\mathbf{x}_{k-1}) - \mathbf{A}\mathbf{r}_{k-1} = \mathbf{r}_{k-1} - \mathbf{A}\mathbf{r}_{k-1} = (\mathbf{I} - \mathbf{A})\mathbf{r}_{k-1}\end{aligned}$$

Non-Stationary/Krylov Subspace Method (2/2)

非定常法・クリロフ部分空間法

$$\mathbf{x}_k = \mathbf{x}_0 + \sum_{i=0}^{k-1} \mathbf{r}_i = \mathbf{x}_0 + \mathbf{r}_0 + \sum_{i=0}^{k-2} (\mathbf{I} - \mathbf{A})\mathbf{r}_i = \mathbf{x}_0 + \mathbf{r}_0 + \sum_{i=1}^{k-1} (\mathbf{I} - \mathbf{A})^i \mathbf{r}_0$$

$$\mathbf{z}_k = \mathbf{r}_0 + \sum_{i=1}^{k-1} (\mathbf{I} - \mathbf{A})^i \mathbf{r}_0 = \left[\mathbf{I} + \sum_{i=1}^{k-1} (\mathbf{I} - \mathbf{A})^i \right] \mathbf{r}_0$$

$\mathbf{z_k}$ is a vector which belongs to kth Krylov Subspace（クリロフ部分空間）, approximate solution vector $\mathbf{x_k}$ is derived by the Krylov Subspace:

$$\left[\mathbf{r}_0, \mathbf{A}\mathbf{r}_0, \mathbf{A}^2\mathbf{r}_0, \ldots, \mathbf{A}^{k-1}\mathbf{r}_0\right]$$

Conjugate Gradient Method
共役勾配法

- Conjugate Gradient: CG
  - Most popular “non-stationary” iterative method
- for Symmetric Positive Definite (SPD) Matrices
  - 対称正定
  - $\{x\}^{\mathrm{T}}[A]\{x\}>0$ for arbitrary $\{x\}$
  - All of diagonal components, eigenvaules and leading principal minors > 0 （主小行列式・首座行列式）
  - Matrices of Galerkin-based FEM: heat conduction, Poisson, static linear elastic problems
- Algorithm
  - “Steepest Descent Method”
  - $\boldsymbol{x^{(i)}= x^{(i-1)} + \alpha_i p^{(i)}}$
    - $x^{(i)}$: solution, $p^{(i)}$: search direction, α_i: coefficient
  - Solution $\{x\}$ minimizes $\boldsymbol{\{x-y\}^{\mathrm{T}}[A]\{x-y\}}$, where $\{y\}$ is exact solution.

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{24} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & a_{34} & \cdots & a_{3n} \\ a_{41} & a_{42} & a_{43} & a_{44} & \cdots & a_{4n} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} & \cdots & a_{nn} \end{bmatrix}$$

Procedures of Conjugate Gradient

Compute $r^{(0)} = b - [A]x^{(0)}$
for i= 1, 2, …
  $z^{(i-1)} = r^{(i-1)}$
  $\rho_{i-1} = r^{(i-1)}\, z^{(i-1)}$
  if i=1
    $p^{(1)} = z^{(0)}$
  else
    $\beta_{i-1} = \rho_{i-1}/\rho_{i-2}$
    $p^{(i)} = z^{(i-1)} + \beta_{i-1}\, p^{(i-1)}$
  endif
  $q^{(i)} = [A]p^{(i)}$
  $\alpha_i = \rho_{i-1}/p^{(i)}q^{(i)}$
  $x^{(i)} = x^{(i-1)} + \alpha_i p^{(i)}$
  $r^{(i)} = r^{(i-1)} - \alpha_i q^{(i)}$
  check convergence $|r|$
end

- Mat-Vec. Multiplication
- Dot Products
- DAXPY (Double Precision: a{X} + {Y})

$x^{(i)}$: Vector

α_i : Scalar

Procedures of Conjugate Gradient

Compute $r^{(0)} = b - [A]x^{(0)}$
for i= 1, 2, …
  $z^{(i-1)} = r^{(i-1)}$
  $\rho_{i-1} = r^{(i-1)}\, z^{(i-1)}$
  if i=1
    $p^{(1)} = z^{(0)}$
  else
    $\beta_{i-1} = \rho_{i-1}/\rho_{i-2}$
    $p^{(i)} = z^{(i-1)} + \beta_{i-1}\, p^{(i-1)}$
  endif
  $q^{(i)} = [A]p^{(i)}$
  $\alpha_i = \rho_{i-1}/p^{(i)}q^{(i)}$
  $x^{(i)} = x^{(i-1)} + \alpha_i p^{(i)}$
  $r^{(i)} = r^{(i-1)} - \alpha_i q^{(i)}$
  check convergence $|r|$
end

- Mat-Vec. Multiplication
- Dot Products
- DAXPY

$x^{(i)}$: Vector

α_i : Scalar

Procedures of Conjugate Gradient

Compute $r^{(0)}= b-[A]x^{(0)}$
for i= 1, 2, …
  $z^{(i-1)}= r^{(i-1)}$
  $\rho_{i-1}= r^{(i-1)}\ z^{(i-1)}$
  if i=1
    $p^{(1)}= z^{(0)}$
  else
    $\beta_{i-1}= \rho_{i-1}/\rho_{i-2}$
    $p^{(i)}= z^{(i-1)} + \beta_{i-1}\ p^{(i-1)}$
  endif
  $q^{(i)}= [A]p^{(i)}$
  $\alpha_i = \rho_{i-1}/p^{(i)}q^{(i)}$
  $x^{(i)}= x^{(i-1)} + \alpha_i p^{(i)}$
  $r^{(i)}= r^{(i-1)} - \alpha_i q^{(i)}$
  check convergence $|r|$
end

- Mat-Vec. Multiplication
- Dot Products
- DAXPY

$x^{(i)}$: Vector

α_i : Scalar

Procedures of Conjugate Gradient

Compute $r^{(0)} = b - [A]x^{(0)}$
for i= 1, 2, …
  $z^{(i-1)} = r^{(i-1)}$
  $\rho_{i-1} = r^{(i-1)}\, z^{(i-1)}$
  if i=1
    $p^{(1)} = z^{(0)}$
  else
    $\beta_{i-1} = \rho_{i-1}/\rho_{i-2}$
    $\mathbf{p^{(i)} = z^{(i-1)} + \beta_{i-1}\, p^{(i-1)}}$
  endif
  $q^{(i)} = [A]p^{(i)}$
  $\alpha_i = \rho_{i-1}/p^{(i)}q^{(i)}$
  $\mathbf{x^{(i)} = x^{(i-1)} + \alpha_i p^{(i)}}$
  $\mathbf{r^{(i)} = r^{(i-1)} - \alpha_i q^{(i)}}$
  check convergence |r|
end

- Mat-Vec. Multiplication
- Dot Products
- DAXPY
  - Double
  - a{x} + {y}

$x^{(i)}$: Vector

α_i : Scalar

Procedures of Conjugate Gradient

Compute $r^{(0)}= b-[A]x^{(0)}$
for i= 1, 2, …
  $z^{(i-1)}= r^{(i-1)}$
  $\rho_{i-1}= r^{(i-1)}\ z^{(i-1)}$
  if i=1
    $p^{(1)}= z^{(0)}$
  else
    $\beta_{i-1}= \rho_{i-1}/\rho_{i-2}$
    $p^{(i)}= z^{(i-1)} + \beta_{i-1}\ p^{(i-1)}$
  endif
  $q^{(i)}= [A]p^{(i)}$
  $\alpha_i = \rho_{i-1}/p^{(i)}q^{(i)}$
  $x^{(i)}= x^{(i-1)} + \alpha_i p^{(i)}$
  $r^{(i)}= r^{(i-1)} - \alpha_i q^{(i)}$
  check convergence $|r|$
end

$x^{(i)}$: Vector
α_i : Scalar

Derivation of CG Algorithm (1/5)

Solution x minimizes the following equation if y is the exact solution ($Ay=b$)

$$(x-y)^T[A](x-y)$$

$$(x-y)^T[A](x-y)=(x,Ax)-(y,Ax)-(x,Ay)+(y,Ay)$$
$$=(x,Ax)-2(x,Ay)+(y,Ay)=(x,Ax)-2(x,b)+\underline{(y,b)}$$ Const.

Therefore, the solution x minimizes the following $f(x)$:

$$f(x)=\frac{1}{2}(x,Ax)-(x,b)$$

$$f(x+h)=f(x)+(h,Ax-b)+\frac{1}{2}(h,Ah)$$ Arbitrary vector h

$$f(x)=\frac{1}{2}(x,Ax)-(x,b)$$

$$f(x+h)=f(x)+(h,Ax-b)+\frac{1}{2}(h,Ah)$$

Arbitrary vector h

$$f(x+h)=\frac{1}{2}(x+h,A(x+h))-(x+h,b)$$

$$=\frac{1}{2}(x+h,Ax)+\frac{1}{2}(x+h,Ah)-(x,b)-(h,b)$$

$$=\frac{1}{2}(x,Ax)+\frac{1}{2}(h,Ax)+\frac{1}{2}(x,Ah)+\frac{1}{2}(h,Ah)-(x,b)-(h,b)$$

$$=\frac{1}{2}(x,Ax)-(x,b)+(h,Ax)-(h,b)+\frac{1}{2}(h,Ah)$$

$$=f(x)+(h,Ax-b)+\frac{1}{2}(h,Ah)$$

Derivation of CG Algorithm (2/5)

CG method minimizes $f(x)$ at each iteration.
Assume that approximate solution: $x^{(k)}$, and
search direction vector $p^{(k)}$ is defined at k-th iteration.

$$x^{(k+1)} = x^{(k)} + \alpha_k p^{(k)}$$

Minimization of $f(x^{(k+1)})$ is done as follows:

$$f\left(x^{(k)} + \alpha_k p^{(k)}\right) = \frac{1}{2}\alpha_k{}^2\left(p^{(k)}, Ap^{(k)}\right) - \alpha_k\left(p^{(k)}, b - Ax^{(k)}\right) + f\left(x^{(k)}\right)$$

$$\frac{\partial f\left(x^{(k)} + \alpha_k p^{(k)}\right)}{\partial \alpha_k} = 0 \Rightarrow \alpha_k = \frac{\left(p^{(k)}, b - Ax^{(k)}\right)}{\left(p^{(k)}, Ap^{(k)}\right)} = \frac{\left(p^{(k)}, r^{(k)}\right)}{\left(p^{(k)}, Ap^{(k)}\right)} \quad \textbf{(1)}$$

$r^{(k)} = b - Ax^{(k)}$ residual vector

Derivation of CG Algorithm (3/5)

Residual vector at $(k+1)$-th iteration:

$$r^{(k+1)} = r^{(k)} - \alpha_k A p^{(k)} \quad (2)$$

$$r^{(k+1)} = b - Ax^{(k+1)},\ r^{(k)} = b - Ax^{(k)}$$

$$r^{(k+1)} - r^{(k)} = -Ax^{(k+1)} + Ax^{(k)} = -\alpha_k A p^{(k)}$$

Search direction vector p is defined by the following recurrence formula:

$$p^{(k+1)} = r^{(k+1)} + \beta_k p^{(k)},\ r^{(0)} = p^{(0)} \quad (3)$$

It's lucky if we can get exact solution y at $(k+1)$-th iteration:

$$y = x^{(k+1)} + \alpha_{k+1} p^{(k+1)}$$

Derivation of CG Algorithm (4/5)

BTW, we have the following (convenient) orthogonality relation:

$$\left(Ap^{(k)}, y - x^{(k+1)}\right) = 0$$

$$\begin{aligned}
&\left(Ap^{(k)}, y - x^{(k+1)}\right) = \left(p^{(k)}, Ay - Ax^{(k+1)}\right) = \left(p^{(k)}, b - Ax^{(k+1)}\right) \\
&= \left(p^{(k)}, b - A\left[x^{(k)} + \alpha_k p^{(k)}\right]\right) = \left(p^{(k)}, b - Ax^{(k)} - \alpha_k Ap^{(k)}\right) \\
&= \left(p^{(k)}, r^{(k)} - \alpha_k Ap^{(k)}\right) = \left(p^{(k)}, r^{(k)}\right) - \alpha_k \left(p^{(k)}, Ap^{(k)}\right) = 0
\end{aligned}$$

$$\because \alpha_k = \frac{\left(p^{(k)}, r^{(k)}\right)}{\left(p^{(k)}, Ap^{(k)}\right)}$$

Thus, following relation is obtained:

$$\left(Ap^{(k)}, y - x^{(k+1)}\right) = \left(Ap^{(k)}, \alpha_{k+1} p^{(k+1)}\right) = 0 \Rightarrow \left(p^{(k+1)}, Ap^{(k)}\right) = 0$$

Derivation of CG Algorithm (5/5)

$$\left(p^{(k+1)}, Ap^{(k)}\right)=\left(r^{(k+1)}+\beta_k p^{(k)}, Ap^{(k)}\right)=\left(r^{(k+1)}, Ap^{(k)}\right)+\beta_k\left(p^{(k)}, Ap^{(k)}\right)=0$$

$$\Rightarrow \beta_k=\frac{-\left(r^{(k+1)}, Ap^{(k)}\right)}{\left(p^{(k)}, Ap^{(k)}\right)} \quad \text{(4)}$$

$\left(p^{(k+1)}, Ap^{(k)}\right)=0$ $p^{(k)}$ and $p^{(k+1)}$ are “conjugate（共役）” for matrix A

```
Compute p(0)=r(0)= b-[A]x(0)
for i= 1, 2, …
    calc. αi-1
    x(i)= x(i-1) + αi-1p(i-1)
    r(i)= r(i-1) - αi-1[A]p(i-1)

    check convergence |r|
    (if not converged)
    calc. βi-1
    p(i)= r(i) + βi-1 p(i-1)
end
```

$$\alpha_{i-1}=\frac{\left(p^{(i-1)}, r^{(i-1)}\right)}{\left(p^{(i-1)}, Ap^{(i-1)}\right)}$$

$$\beta_{i-1}=\frac{-\left(r^{(i)}, Ap^{(i-1)}\right)}{\left(p^{(i-1)}, Ap^{(i-1)}\right)}$$

Properties of CG Algorithm

Following “conjugate（共役）” relationship is obtained for arbitrary (i,j):

$$\left(p^{(i)}, Ap^{(j)}\right) = 0 \ (i \neq j)$$

Following relationships are also obtained for $p^{(k)}$ and $r^{(k)}$:

$$\left(r^{(i)}, r^{(j)}\right) = 0 \ (i \neq j), \quad \left(p^{(k)}, r^{(k)}\right) = \left(r^{(k)}, r^{(k)}\right)$$

In N-dimensional space, only N sets of orthogonal and linearly independent residual vector $r^{(k)}$. This means CG method converges after N iterations if number of unknowns is N. Actually, round-off error sometimes affects convergence.

Proof (1/3)

Mathematical Induction

数学的帰納法

$$\left(r^{(i)}, r^{(j)}\right) = 0 \left(i \neq j\right)$$
$$\left(p^{(i)}, Ap^{(j)}\right) = 0 \left(i \neq j\right)$$

(1) $$\alpha_k = \frac{\left(p^{(k)}, r^{(k)}\right)}{\left(p^{(k)}, Ap^{(k)}\right)}$$

(2) $$r^{(k+1)} = r^{(k)} - \alpha_k Ap^{(k)}$$

(3) $$p^{(k+1)} = r^{(k+1)} + \beta_k p^{(k)}, \; r^{(0)} = p^{(0)}$$

(4) $$\beta_k = \frac{-\left(r^{(k+1)}, Ap^{(k)}\right)}{\left(p^{(k)}, Ap^{(k)}\right)}$$

Proof (2/3)

Mathematical Induction
数学的帰納法

$$\left(r^{(i)}, r^{(j)}\right) = 0 \ (i \neq j)$$
$$\left(p^{(i)}, Ap^{(j)}\right) = 0 \ (i \neq j) \qquad (*)$$

$(*)$ is satisfied for $i \leq k,\ j \leq k$ where $i \neq j$

$$\underline{\text{if } i < k} \quad \left(r^{(k+1)}, r^{(i)}\right) = \left(r^{(i)}, r^{(k+1)}\right) \overset{(2)}{=} \left(r^{(i)}, r^{(k)} - \alpha_k Ap^{(k)}\right)$$
$$\overset{(*)}{=} -\alpha_k \left(r^{(i)}, Ap^{(k)}\right) \overset{(3)}{=} -\alpha_k \left(p^{(i)} - \beta_{i-1} p^{(i-1)}, Ap^{(k)}\right)$$
$$= -\alpha_k \left(p^{(i)}, Ap^{(k)}\right) + \alpha_k \beta_{i-1} \left(p^{(i-1)}, Ap^{(k)}\right) \overset{(*)}{=} 0$$

$$\underline{\text{if } i = k} \quad \left(r^{(k+1)}, r^{(k)}\right) \overset{(2)}{=} \left(r^{(k)}, r^{(k)}\right) - \left(r^{(k)}, \alpha_k Ap^{(k)}\right)$$
$$\overset{(3)}{=} \left(r^{(k)}, r^{(k)}\right) - \left(p^{(k)} - \beta_{k-1} p^{(k-1)}, \alpha_k Ap^{(k)}\right)$$
$$\overset{(*)}{=} \left(r^{(k)}, r^{(k)}\right) - \alpha_k \left(p^{(k)}, Ap^{(k)}\right) \overset{(1)}{=} \left(r^{(k)}, r^{(k)}\right) - \left(p^{(k)}, r^{(k)}\right)$$
$$\overset{(3)}{=} \left(r^{(k)}, r^{(k)}\right) - \left(\beta_{k-1} p^{(k-1)} + r^{(k)}, r^{(k)}\right)$$
$$= -\beta_{k-1} \left(p^{(k-1)}, r^{(k)}\right) \overset{(2)}{=} -\beta_{k-1} \left(p^{(k-1)}, r^{(k-1)} - \alpha_{k-1} Ap^{(k-1)}\right)$$
$$= -\beta_{k-1} \left\{\left(p^{(k-1)}, r^{(k-1)}\right) - \alpha_{k-1} \left(p^{(k-1)}, Ap^{(k-1)}\right)\right\} \overset{(1)}{=} 0$$

(1) $\alpha_k = \dfrac{\left(p^{(k)}, r^{(k)}\right)}{\left(p^{(k)}, Ap^{(k)}\right)}$

(2) $r^{(k+1)} = r^{(k)} - \alpha_k Ap^{(k)}$

(3) $p^{(k+1)} = r^{(k+1)} + \beta_k p^{(k)}$

(4) $\beta_k = \dfrac{-\left(r^{(k+1)}, Ap^{(k)}\right)}{\left(p^{(k)}, Ap^{(k)}\right)}$

Proof (3/3)

Mathematical Induction
数学的帰納法

$$\left(r^{(i)}, r^{(j)}\right) = 0 \ (i \neq j)$$
$$\left(p^{(i)}, Ap^{(j)}\right) = 0 \ (i \neq j) \quad (*)$$

(*) is satisfied for $i \leq k,\ j \leq k$ where $i \neq j$

(1) $\alpha_k = \dfrac{\left(p^{(k)}, r^{(k)}\right)}{\left(p^{(k)}, Ap^{(k)}\right)}$

(2) $r^{(k+1)} = r^{(k)} - \alpha_k Ap^{(k)}$

(3) $p^{(k+1)} = r^{(k+1)} + \beta_k p^{(k)}$

(4) $\beta_k = \dfrac{-\left(r^{(k+1)}, Ap^{(k)}\right)}{\left(p^{(k)}, Ap^{(k)}\right)}$

if $i < k$

$$\left(p^{(k+1)}, Ap^{(i)}\right) \overset{(3)}{=} \left(r^{(k+1)} + \beta_k p^{(k)}, Ap^{(i)}\right)$$
$$\overset{(*)}{=} \left(r^{(k+1)}, Ap^{(i)}\right)$$
$$\overset{(2)}{=} \frac{1}{\alpha_i}\left(r^{(k+1)}, r^{(i)} - r^{(i+1)}\right) = 0$$

if $i = k$

$$\left(p^{(k+1)}, Ap^{(k)}\right) \overset{(3)}{=} \left(r^{(k+1)}, Ap^{(k)}\right) + \beta_k \left(p^{(k)}, Ap^{(k)}\right)$$
$$\overset{(4)}{=} 0$$

$$\left(r^{(k+1)}, r^{(k)}\right) = 0$$

$$\left(r^{(k+1)}, r^{(k)}\right) \overset{(2)}{=} \left(r^{(k)}, r^{(k)}\right) - \left(r^{(k)}, \alpha_k A p^{(k)}\right)$$

$$\overset{(3)}{=} \left(r^{(k)}, r^{(k)}\right) - \left(p^{(k)} - \beta_{k-1} p^{(k-1)}, \alpha_k A p^{(k)}\right)$$

$$\overset{(*)}{=} \left(r^{(k)}, r^{(k)}\right) - \alpha_k \left(p^{(k)}, A p^{(k)}\right) \overset{(1)}{=} \left(r^{(k)}, r^{(k)}\right) - \left(p^{(k)}, r^{(k)}\right) = 0$$

$$\therefore \left(r^{(k)}, r^{(k)}\right) = \left(p^{(k)}, r^{(k)}\right)$$

(1) $\alpha_k = \dfrac{\left(p^{(k)}, r^{(k)}\right)}{\left(p^{(k)}, A p^{(k)}\right)}$

(2) $r^{(k+1)} = r^{(k)} - \alpha_k A p^{(k)}$

(3) $p^{(k+1)} = r^{(k+1)} + \beta_k p^{(k)}$

(4) $\beta_k = \dfrac{-\left(r^{(k+1)}, A p^{(k)}\right)}{\left(p^{(k)}, A p^{(k)}\right)}$

$\alpha_k, \quad \beta_k$

Usually, we use simpler definitions of α_k, β_k as follows:

$$\alpha_k = \frac{\left(p^{(k)}, b - Ax^{(k)}\right)}{\left(p^{(k)}, Ap^{(k)}\right)} = \frac{\left(p^{(k)}, r^{(k)}\right)}{\left(p^{(k)}, Ap^{(k)}\right)} = \frac{\left(r^{(k)}, r^{(k)}\right)}{\left(p^{(k)}, Ap^{(k)}\right)}$$

$$\because \left(p^{(k)}, r^{(k)}\right) = \left(r^{(k)}, r^{(k)}\right)$$

$$\beta_k = \frac{-\left(r^{(k+1)}, Ap^{(k)}\right)}{\left(p^{(k)}, Ap^{(k)}\right)} = \frac{\left(r^{(k+1)}, r^{(k+1)}\right)}{\left(r^{(k)}, r^{(k)}\right)}$$

$$\because \left(r^{(k+1)}, Ap^{(k)}\right) = \frac{\left(r^{(k+1)}, r^{(k)} - r^{(k+1)}\right)}{\alpha_k} = -\frac{\left(r^{(k+1)}, r^{(k+1)}\right)}{\alpha_k}$$

Procedures of Conjugate Gradient

```
Compute r(0)= b-[A]x(0)
for i= 1, 2, …
    z(i-1)= r(i-1)
    ρi-1= r(i-1) z(i-1)
    if i=1
      p(1)= z(0)
     else
      βi-1= ρi-1/ρi-2
      p(i)= z(i-1) + βi-1 p(i-1)
    endif
    q(i)= [A]p(i)
    αi  = ρi-1/p(i)q(i)
    x(i)= x(i-1) + αip(i)
    r(i)= r(i-1) - αiq(i)
    check convergence |r|
end
```

$x^{(i)}$: Vector

α_i : Scalar

$$\beta_{i-1} = \frac{\left(r^{(i-1)}, r^{(i-1)}\right)}{\left(r^{(i-2)}, r^{(i-2)}\right)} \quad \left(= \rho_{i-1}\right) \quad \left(= \rho_{i-2}\right)$$

$$\alpha_i = \frac{\left(r^{(i-1)}, r^{(i-1)}\right)}{\left(p^{(i)}, Ap^{(i)}\right)} \quad \left(= \rho_{i-1}\right)$$

Preconditioning for Iterative Solvers

- Convergence rate of iterative solvers strongly depends on the spectral properties (eigenvalue distribution) of the coefficient matrix **A**.
  - Eigenvalue distribution is small, eigenvalues are close to 1
  - In "ill-conditioned" problems, "condition number" (ratio of max/min eigenvalue if **A** is symmetric) is large（条件数）.
- A preconditioner **M** (whose properties are similar to those of **A**) transforms the linear system into one with more favorable spectral properties （前処理）
  - **M** transforms **Ax=b** into **A'x=b'** where $\mathbf{A'=M^{-1}A}$, $\mathbf{b'=M^{-1}b}$
  - If **M~A**, $\mathbf{M^{-1}A}$ is close to identity matrix.
  - If $\mathbf{M^{-1}=A^{-1}}$, this is the best preconditioner (Gaussian Elim.)
  - Generally, **A'x'=b'** where $\mathbf{A'=M_L^{-1}AM_R^{-1}}$, $\mathbf{b'=M_L^{-1}b}$, $\mathbf{x'=M_Rx}$
  - $\mathbf{M_L}$/$\mathbf{M_R}$: Left/Right Preconditioning（左／右前処理）

Preconditioned CG Solver

Compute $r^{(0)}= b-[A]x^{(0)}$
for i= 1, 2, …
  solve $[M]z^{(i-1)}= r^{(i-1)}$
  $\rho_{i-1}= r^{(i-1)}\ z^{(i-1)}$
  if i=1
    $p^{(1)}= z^{(0)}$
  else
    $\beta_{i-1}= \rho_{i-1}/\rho_{i-2}$
    $p^{(i)}= z^{(i-1)} + \beta_{i-1}\ p^{(i-1)}$
  endif
  $q^{(i)}= [A]p^{(i)}$
  $\alpha_i = \rho_{i-1}/p^{(i)}q^{(i)}$
  $x^{(i)}= x^{(i-1)} + \alpha_i p^{(i)}$
  $r^{(i)}= r^{(i-1)} - \alpha_i q^{(i)}$
  check convergence $|r|$
end

$$[M]= [M_1][M_2]$$

$$[A']x'=b'$$
$$[A']=[M_1]^{-1}[A][M_2]^{-1}$$
$$x'=[M_2]x,\ b'=[M_1]^{-1}b$$

$$p'=>[M_2]p,\ r'=>[M_1]^{-1}r$$

$$\overset{x}{p}'^{(i)}= r'^{(i-1)} + \beta'_{i-1}\ p'^{(i-1)}$$

$$[M_2]p^{(i)}= [M_1]^{-1}r^{(i-1)} + \beta'_{i-1}\ [M_2]p^{(i-1)}$$

$$p^{(i)}= [M_2]^{-1}[M_1]^{-1}r^{(i-1)} + \beta'_{i-1}\ p^{(i-1)}$$
$$p^{(i)}= [M]^{-1}r^{(i-1)} + \beta'_{i-1}\ p^{(i-1)}$$

$$\beta'_{i-1}= ([M]^{-1}r^{(i-1)}, r^{(i-1)})/([M]^{-1}r^{(i-2)}, r^{(i-2)})$$
$$\alpha'_{i-1}= ([M]^{-1}r^{(i-1)}, r^{(i-1)})/(p^{(i-1)}, [A]p^{(i-1)})$$

In CG method, preconditioner usually satisfies $[M_2]=[M_1]^T$, such as Incomplete Cholesky/Incomplete Modified Cholesky Factorizations. In this problem, let us define $[M_1]$ and $[M_2]$ as follows:

$$[M_1]=[X]^T,[M_2]=[X],[M]=[M_1][M_2]$$

$$[A']x'=b'$$

$$[A']=[M_1]^{-1}[A][M_2]^{-1}=[[X]^T]^{-1}[A][X]^{-1}=[X]^{-T}[A][X]^{-1}$$

$$x'=[X]x,\quad b'=[X]^{-T}b,\quad r'=[X]^{-T}r$$

$$\alpha'_{i-1}=\frac{\left(r'^{(i-1)},r'^{(i-1)}\right)}{\left(p'^{(i-1)},A'p'^{(i-1)}\right)}=\frac{\left([X]^{-T}r^{(i-1)},[X]^{-T}r^{(i-1)}\right)}{\left([X]p^{(i-1)},[X]^{-T}[A][X]^{-1}[X]p^{(i-1)}\right)}$$

$$=\frac{\left(\left([X]^{-T}r^{(i-1)}\right)^T,[X]^{-T}r^{(i-1)}\right)}{\left(\left([X]p^{(i-1)}\right)^T,[X]^{-T}[A]p^{(i-1)}\right)}=\frac{\left(\left(r^{(i-1)}\right)^T[X]^{-1},\left[X^T\right]^{-1}r^{(i-1)}\right)}{\left(\left(p^{(i-1)}\right)^T[X]^T,[X]^{-T}[A]p^{(i-1)}\right)}$$

$$=\frac{\left(r^{(i-1)},\left[\left[X^T\right][X]\right]^{-1}r^{(i-1)}\right)}{\left(p^{(i-1)},[A]p^{(i-1)}\right)}=\frac{\left(r^{(i-1)},[M]^{-1}r^{(i-1)}\right)}{\left(p^{(i-1)},[A]p^{(i-1)}\right)}=\frac{\left(r^{(i-1)},z^{(i-1)}\right)}{\left(p^{(i-1)},[A]p^{(i-1)}\right)}$$

$$\beta'_{i-1} = \frac{\left(r'^{(i-1)}, r'^{(i-1)}\right)}{\left(r'^{(i-2)}, r'^{(i-2)}\right)} == \frac{\left(\left[\mathrm{X}\right]^{-T} r^{(i-1)}, \left[\mathrm{X}\right]^{-T} r^{(i-1)}\right)}{\left(\left[\mathrm{X}\right]^{-T} r^{(i-2)}, \left[\mathrm{X}\right]^{-T} r^{(i-2)}\right)}$$

$$= \frac{\left(\left(\left[\mathrm{X}\right]^{-T} r^{(i-1)}\right)^{T}, \left[\mathrm{X}\right]^{-T} r^{(i-1)}\right)}{\left(\left(\left[\mathrm{X}\right]^{-T} r^{(i-2)}\right)^{T}, \left[\mathrm{X}\right]^{-T} r^{(i-2)}\right)} = \frac{\left(\left(r^{(i-1)}\right)^{T}\left[\mathrm{X}\right]^{-1}, \left[\mathrm{X}^{T}\right]^{-1} r^{(i-1)}\right)}{\left(\left(r^{(i-2)}\right)^{T}\left[\mathrm{X}\right]^{-1}, \left[\mathrm{X}^{T}\right]^{-1} r^{(i-2)}\right)}$$

$$= \frac{\left(r^{(i-1)}, \left[\left[\mathrm{X}^{T}\right]\left[\mathrm{X}\right]\right]^{-1} r^{(i-1)}\right)}{\left(r^{(i-2)}, \left[\left[\mathrm{X}^{T}\right]\left[\mathrm{X}\right]\right]^{-1} r^{(i-2)}\right)} = \frac{\left(r^{(i-1)}, \left[\mathrm{M}\right]^{-1} r^{(i-1)}\right)}{\left(r^{(i-2)}, \left[\mathrm{M}\right]^{-1} r^{(i-2)}\right)} = \frac{\left(r^{(i-1)}, z^{(i-1)}\right)}{\left(r^{(i-2)}, z^{(i-2)}\right)}$$

Preconditioned Conjugate Gradient Method （PCG）

```
Compute r(0)= b-[A]x(0)
for i= 1, 2, …
    solve [M]z(i-1)= r(i-1)
    ρi-1= r(i-1) z(i-1)
    if i=1
      p(1)= z(0)
     else
      βi-1= ρi-1/ρi-2
      p(i)= z(i-1) + βi-1 p(i-1)
    endif
    q(i)= [A]p(i)
    αi  = ρi-1/p(i)q(i)
    x(i)= x(i-1) + αip(i)
    r(i)= r(i-1) – αiq(i)
    check convergence |r|
end
```

Solving the following equation:

$$\{z\}=[M]^{-1}\{r\}$$

“Approximate Inverse Matrix”

$$[M]^{-1}\approx[A]^{-1},\quad [M]\approx[A]$$

Ultimate Preconditioning: Inverse Matrix

$$[M]^{-1}=[A]^{-1},\quad [M]=[A]$$

Diagonal Scaling: Simple but weak

$$[M]^{-1}=[D]^{-1},\quad [M]=[D]$$

ILU(0), IC(0)

- Widely used Preconditioners for Sparse Matrices
  - Incomplete LU Factorization（不完全LU分解）
  - Incomplete Cholesky Factorization (for Symmetric Matrices)（不完全コレスキー分解）

- Incomplete Direct Method
  - Even if original matrix is sparse, inverse matrix is not necessarily sparse.
  - fill-in
  - ILU(0)/IC(0) without fill-in have same non-zero pattern with the original (sparse) matrices

Diagonal Scaling, Point-Jacobi

$$[M] = \begin{bmatrix} D_1 & 0 & \dots & 0 & 0 \\ 0 & D_2 & & 0 & 0 \\ \dots & & \dots & & \dots \\ 0 & 0 & & D_{N-1} & 0 \\ 0 & 0 & \dots & 0 & D_N \end{bmatrix}$$

- **`solve [M]z(i-1)= r(i-1)`** is very easy.
- Provides fast convergence for simple problems.
- 1d.f, 1d.c

- 1D-code for Static Linear-Elastic Problems by Galerkin FEM
- Sparse Linear Solver
  - Conjugate Gradient Method
  - Preconditioning
- Storage of Sparse Matrices

- Program

Coef. Matrix derived from FEM

- Sparse Matrix
  - Many “0”’s
- Storing all components (e.g. A(i,j)) is not efficient for sparse matrices
  - A(i,j) is suitable for dense matrices

$$
\begin{bmatrix}
D & X & & & X & X & & & & & & & & & & \\
X & D & X & & X & X & X & & & & & & & & & \\
 & X & D & X & & X & X & X & & & & & & & & \\
 & & X & D & & & X & X & & & & & & & & \\
X & X & & & D & X & & & X & X & & & & & & \\
X & X & X & & X & D & X & & X & X & X & & & & & \\
 & X & X & X & & X & D & X & & X & X & X & & & & \\
 & & X & X & & & X & D & & & X & X & & & & \\
 & & & & X & X & & & D & X & & & X & X & & \\
 & & & & X & X & X & & X & D & X & & X & X & X & \\
 & & & & & X & X & X & & X & D & X & & X & X & X \\
 & & & & & & X & X & & & X & D & & & X & X \\
 & & & & & & & & X & X & & & D & X & & \\
 & & & & & & & & X & X & X & & X & D & X & \\
 & & & & & & & & & X & X & & & X & D & X \\
 & & & & & & & & & & X & X & & & X & D
\end{bmatrix}
\begin{Bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \Phi_4 \\ \Phi_5 \\ \Phi_6 \\ \Phi_7 \\ \Phi_8 \\ \Phi_9 \\ \Phi_{10} \\ \Phi_{11} \\ \Phi_{12} \\ \Phi_{13} \\ \Phi_{14} \\ \Phi_{15} \\ \Phi_{16} \end{Bmatrix}
=
\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \\ F_9 \\ F_{10} \\ F_{11} \\ F_{12} \\ F_{13} \\ F_{14} \\ F_{15} \\ F_{16} \end{Bmatrix}
$$

- Number of non-zero off-diagonal components is O(100) in FEM
  - If number of unknowns is 10^8 :
    - A(i,j): $O(10^{16})$ words ~ $O(10^{17})$ bytes for DP: 100PB (400x Odyssey)
    - Actual Non-zero Comp. : $O(10^{10})$ words: 100GB (Odyssey: 32GB/node)
- Only (really) non-zero off-diag. components should be stored on memory

Variables/Arrays in 1d.f, 1d.c related to coefficient matrix

name	type	size	description
N	I	–	# Unknowns
NPLU	I	–	# Non-Zero Off-Diagonal Components
Diag(:)	R	N	Diagonal Components
PHI(:)	R	N	Unknown Vector
Rhs(:)	R	N	RHS Vector
Index(:)	I	0:N N+1	Off-Diagonal Components (Number of Non-Zero Off-Diagonals at Each ROW)
Item(:)	I	NPLU	Off-Diagonal Components (Corresponding Column ID)
AMat(:)	R	NPLU	Off-Diagonal Components (Value)

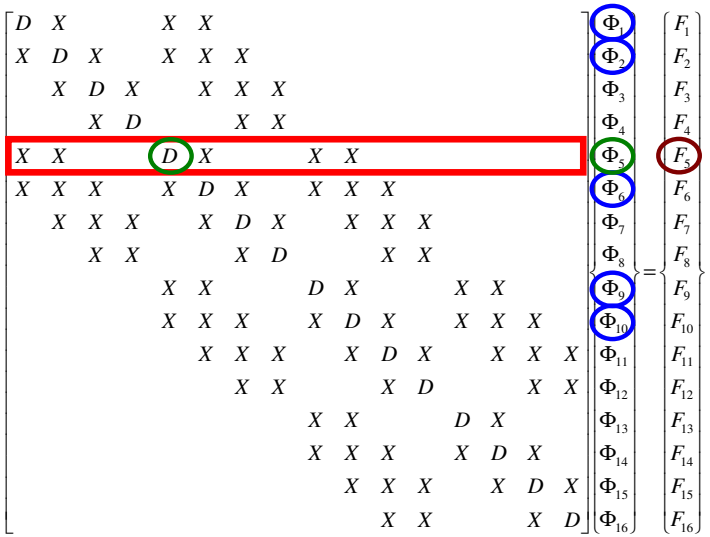
Only non-zero components are stored according to “Compressed Row Storage”.

Mat-Vec. Multiplication for Sparse Matrix

Compressed Row Storage (CRS)

Diag (i)	Diagonal Components (REAL, i=1~N)
Index(i)	Number of Non-Zero Off-Diagonals at Each ROW (INT, i=0~N)
Item(k)	Off-Diagonal Components (Corresponding Column ID) （INT, k=1, index(N)）
AMat(k)	Off-Diagonal Components (Value) （ REAL, k=1, index(N) ）

```
{Y}=[A]{X}

for(i=0;i<N;i++){
        Y[i] = Diag[i] * X[i];
        for(k=Index[i];k<Index[i+1];k++){
                Y[i] += AMat[k]*X[Item[k]];
        }
}
```



Mat-Vec. Multiplication for Dense Matrix

Very Easy, Straightforward

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1,N-1} & a_{1,N} \\ a_{21} & a_{22} & & a_{2,N-1} & a_{2,N} \\ \cdots & & \cdots & & \cdots \\ a_{N-1,1} & a_{N-1,2} & & a_{N-1,N-1} & a_{N-1,N} \\ a_{N,1} & a_{N,2} & \cdots & a_{N,N-1} & a_{N,N} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{N-1} \\ x_N \end{Bmatrix} = \begin{Bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{N-1} \\ y_N \end{Bmatrix}$$

```
{Y}= [A]{X}

for (i=0;i<N;j++){
  Y[i]= 0.d0;
  for (j=0;j<N;j++){
    Y[i]= Y[i] + A[i][j]*X[j]
  }
}
```

Compressed Row Storage (CRS)

	1	2	3	4	5	6	7	8
1	1.1	2.4	0	0	3.2	0	0	0
2	4.3	3.6	0	2.5	0	3.7	0	9.1
3	0	0	5.7	0	1.5	0	3.1	0
4	0	4.1	0	9.8	2.5	2.7	0	0
5	3.1	9.5	10.4	0	11.5	0	4.3	0
6	0	0	6.5	0	0	12.4	9.5	0
7	0	6.4	2.5	0	0	1.4	23.1	13.1
8	0	9.5	1.3	9.6	0	3.1	0	51.3

Compressed Row Storage (CRS): C Numbering starts from 0 in program

	0	1	2	3	4	5	6	7
0	**1.1** ◎	2.4 ①			3.2 ④			
1	4.3 ◎	**3.6** ①		2.5 ③		3.7 ⑤		9.1 ⑦
2			**5.7** ②		1.5 ④		3.1 ⑥	
3		4.1 ①		**9.8** ③	2.5 ④	2.7 ⑤		
4	3.1 ◎	9.5 ①	10.4 ②		**11.5** ④		4.3 ⑥	
5			6.5 ②			**12.4** ⑤	9.5 ⑥	
6		6.4 ①	2.5 ②			1.4 ⑤	**23.1** ⑥	13.1 ⑦
7		9.5 ①	1.3 ②	9.6 ③		3.1 ⑤		**51.3** ⑦

N= 8

Diagonal Components

```
Diag[0]=  1.1
Diag[1]=  3.6
Diag[2]=  5.7
Diag[3]=  9.8
Diag[4]= 11.5
Diag[5]= 12.4
Diag[6]= 23.1
Diag[7]= 51.3
```

Compressed Row Storage (CRS) : C

	0	1	2	3	4	5	6	7
0	1.1 ◎		2.4 ①			3.2 ④			
1	3.6 ①	4.3 ◎			2.5 ③		3.7 ⑤		9.1 ⑦
2	5.7 ②					1.5 ④		3.1 ⑥	
3	9.8 ③		4.1 ①			2.5 ④	2.7 ⑤		
4	11.5 ④	3.1 ◎	9.5 ①	10.4 ②				4.3 ⑥	
5	12.4 ⑤			6.5 ②				9.5 ⑥	
6	23.1 ⑥		6.4 ①	2.5 ②			1.4 ⑤		13.1 ⑦
7	51.3 ⑦		9.5 ①	1.3 ②	9.6 ③		3.1 ⑤		

Compressed Row Storage (CRS) : C

						# Non-Zero Off-Diag.	
							Index[0]= 0
0	1.1 ◎	2.4 ①	3.2 ④			2	Index[1]= 2
1	3.6 ①	4.3 ◎	2.5 ③	3.7 ⑤	9.1 ⑦	4	Index[2]= 6
2	5.7 ②	1.5 ④	3.1 ⑥			2	Index[3]= 8
3	9.8 ③	4.1 ①	2.5 ④	2.7 ⑤		3	Index[4]= 11
4	11.5 ④	3.1 ◎	9.5 ①	10.4 ②	4.3 ⑥	4	Index[5]= 15
5	12.4 ⑤	6.5 ②	9.5 ⑥			2	Index[6]= 17
6	23.1 ⑥	6.4 ①	2.5 ②	1.4 ⑤	13.1 ⑦	4	Index[7]= 21
7	51.3 ⑦	9.5 ①	1.3 ②	9.6 ③	3.1 ⑤	4	Index[8]= 25

NPLU= 25 (=Index[N])

(Index[i])th~(Index[i+1])th:
Non-Zero Off-Diag. Components corresponding to i-th row.

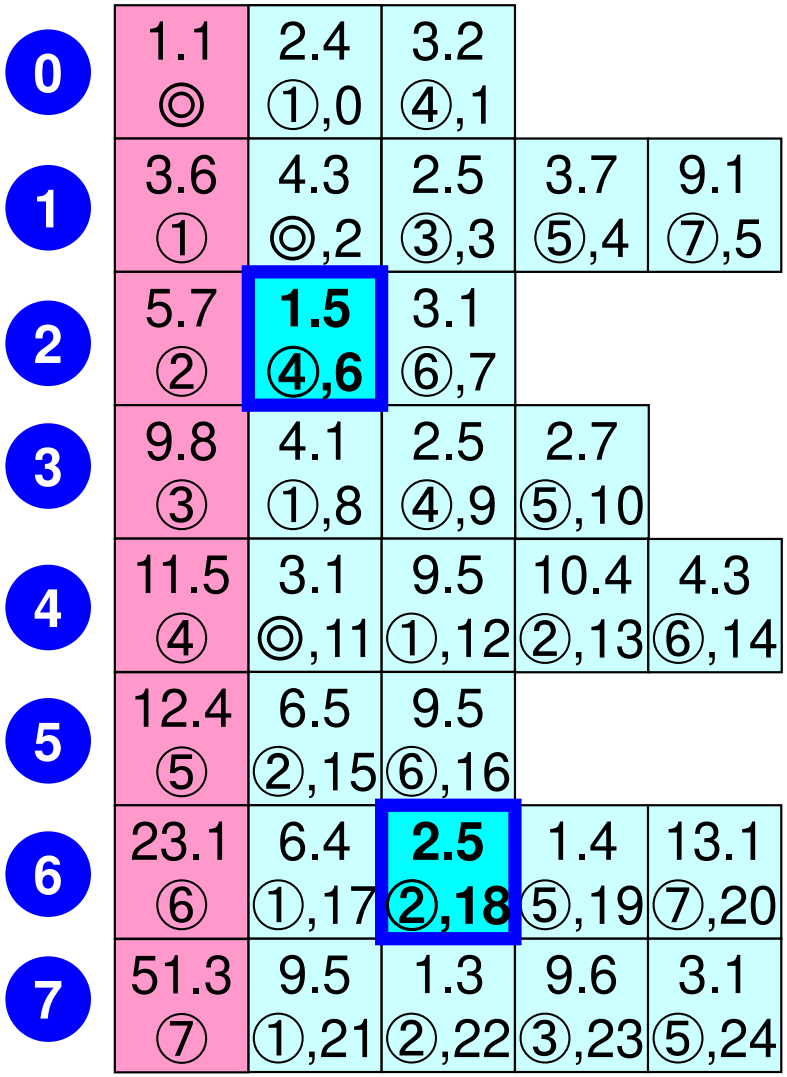
Compressed Row Storage (CRS) : C

						# Non-Zero Off-Diag.	
							Index[0]= 0
0	1.1 ◎	2.4 ①,0	3.2 ④,1			2	Index[1]= 2
1	3.6 ①	4.3 ◎,2	2.5 ③,3	3.7 ⑤,4	9.1 ⑦,5	4	Index[2]= 6
2	5.7 ②	1.5 ④,6	3.1 ⑥,7			2	**Index[3]= 8**
3	9.8 ③	**4.1 ①,8**	**2.5 ④,9**	**2.7 ⑤,10**		**3**	**Index[4]= 11**
4	11.5 ④	3.1 ◎,11	9.5 ①,12	10.4 ②,13	4.3 ⑥,14	4	Index[5]= 15
5	12.4 ⑤	6.5 ②,15	9.5 ⑥,16			2	Index[6]= 17
6	23.1 ⑥	6.4 ①,17	2.5 ②,18	1.4 ⑤,19	13.1 ⑦,20	4	Index[7]= 21
7	51.3 ⑦	9.5 ①,21	1.3 ②,22	9.6 ③,23	3.1 ⑤,24	4	Index[8]= 25

NPLU= 25 (=Index[N])

(Index[i])th~(Index[i+1])th:
Non-Zero Off-Diag. Components corresponding to i-th row.

Compressed Row Storage (CRS) : C




```
Item[ 6]= 4, AMat[ 6]= 1.5
Item[18]= 2, AMat[18]= 2.5
```

Compressed Row Storage (CRS) : C

0	1.1 ◎	2.4 ①,0	3.2 ④,1		
1	3.6 ①	4.3 ◎,2	2.5 ③,3	3.7 ⑤,4	9.1 ⑦,5
2	5.7 ②	1.5 ④,6	3.1 ⑥,7		
3	9.8 ③	4.1 ①,8	2.5 ④,9	2.7 ⑤,10	
4	11.5 ④	3.1 ◎,11	9.5 ①,12	10.4 ②,13	4.3 ⑥,14
5	12.4 ⑤	6.5 ②,15	9.5 ⑥,16		
6	23.1 ⑥	6.4 ①,17	2.5 ②,18	1.4 ⑤,19	13.1 ⑦,20
7	51.3 ⑦	9.5 ①,21	1.3 ②,22	9.6 ③,23	3.1 ⑤,24

Diag [i]	Diagonal Components (REAL, i=0~N-1)
Index[i[	Number of Non-Zero Off-Diagonals at Each ROW (INT, i=0~N)
Item[k]	Off-Diagonal Components (Corresponding Column ID) (INT, k=0, index[N])
Amat[k]	Off-Diagonal Components (Value) (REAL, k=0, index[N])

```
{Y}=[A]{X}

for(i=0;i<N;i++){
    Y[i] = Diag[i] * X[i];
    for(k=Index[i];k<Index[i+1];k++){
        Y[i] += AMat[k]*X[Item[k]];
      }
}
```

- 1D-code for Static Linear-Elastic Problems by Galerkin FEM
- Sparse Linear Solver
  - Conjugate Gradient Method
  - Preconditioning
- Storage of Sparse Matrices

- Program

Finite Element Procedures

- Initialization
  - Control Data
  - Node, Connectivity of Elements (N: Node#, NE: Elem#)
  - Initialization of Arrays (Global/Element Matrices)
  - Element-Global Matrix Mapping (Index, Item)
- Generation of Matrix
  - Element-by-Element Operations (do icel= 1, NE)
    - Element matrices
    - Accumulation to global matrix
  - Boundary Conditions
- Linear Solver
  - Conjugate Gradient Method

Program: 1d.c (1/6)
variables and arrays

```
/*
//   1D Steady-State Heat Transfer
//   FEM with Piece-wise Linear Elements
//   CG (Conjugate Gradient) Method
//
//   d/dx(CdT/dx) + Q = 0
//   T=0@x=0
*/
#include <stdio.h>
#include <stdlib.h>
#include <math.h>
#include <assert.h>

int main(){
        int NE, N, NPLU, IterMax;
        int R, Z, Q, P, DD;

        double dX, Resid, Eps, Area, QV, COND;
        double X1, X2, U1, U2, DL, Strain, Sigma, Ck;
        double QN, XL, C2, Xi, PHIa;
        double *PHI, *Rhs, *X;
        double *Diag, *AMat;
        double **W;
        int *Index, *Item, *Icelnod;
        double Kmat[2][2], Emat[2][2];
        int i, j, in1, in2, k, icel, k1, k2, jS;
        int iter;
        FILE *fp;
        double BNorm2, Rho, Rho1=0.0, C1, Alpha, DNorm2;
        int ierr  = 1;
        int errno = 0;
```

Variable/Arrays (1/2)

Name	Type	Size	I/O	Definition
NE	I		I	# Element
N	I		O	# Node
NPLU	I		O	# Non-Zero Off-Diag. Components
IterMax	I		I	MAX Iteration Number for CG
errno	I		O	ERROR flag
R, Z, Q, P, DD	I		O	Name of Vectors in CG
dX	R		I	Length of Each Element
Resid	R		O	Residual for CG
Eps	R		I	Convergence Criteria for CG
Area	R		I	Sectional Area of Element
QV	R		I	Heat Generation Rate/Volume/Time $\dot{Q}$
COND	R		I	Thermal Conductivity

Variable/Arrays (2/2)

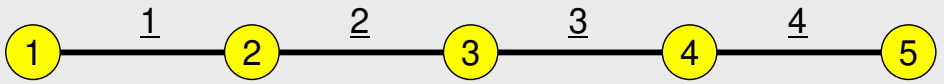
Name	Type	Size	I/O	Definition
X	R	N	O	Location of Each Node
PHI	R	N	O	Temperature of Each Node
Rhs	R	N	O	RHS Vector
Diag	R	N	O	Diagonal Components
W	R	[4][N]	O	Work Array for CG
Amat	R	NPLU	O	Off-Diagonal Components (Value)
Index	I	N+1	O	Number of Non-Zero Off-Diagonals at Each ROW
Item	I	NPLU	O	Off-Diagonal Components (Corresponding Column ID)
Icelnod	I	2*NE	O	Node ID for Each Element
Kmat	R	[2][2]	O	Element Matrix [k]
Emat	R	[2][2]	O	Element Matrix

Program: 1d.c (2/6)
Initialization, Allocation of Arrays

```
/*
// +-------+
// | INIT. |
// +-------+
*/
      fp = fopen("input.dat", "r");
      assert(fp != NULL);
      fscanf(fp, "%d", &NE);
      fscanf(fp, "%lf %lf %lf %lf", &dX, &QV, &Area, &COND);
      fscanf(fp, "%d", &IterMax);
      fscanf(fp, "%lf", &Eps);
      fclose(fp);

      N= NE + 1;

      PHI  = calloc(N, sizeof(double));
      X    = calloc(N, sizeof(double));
      Diag = calloc(N, sizeof(double));

      AMat = calloc(2*N-2, sizeof(double));

      Rhs = calloc(N, sizeof(double));

      Index= calloc(N+1, sizeof(int));
      Item = calloc(2*N-2, sizeof(int));

      Icelnod= calloc(2*NE, sizeof(int));
```

Control Data `input.dat`

```
4                    NE (Number of Elements)
1.0 1.0 1.0 1.0      Δx (Length of Each Elem.: L), Q, A, λ
100                  Number of MAX. Iterations for CG Solver
1.e-8                Convergence Criteria for CG Solver
```



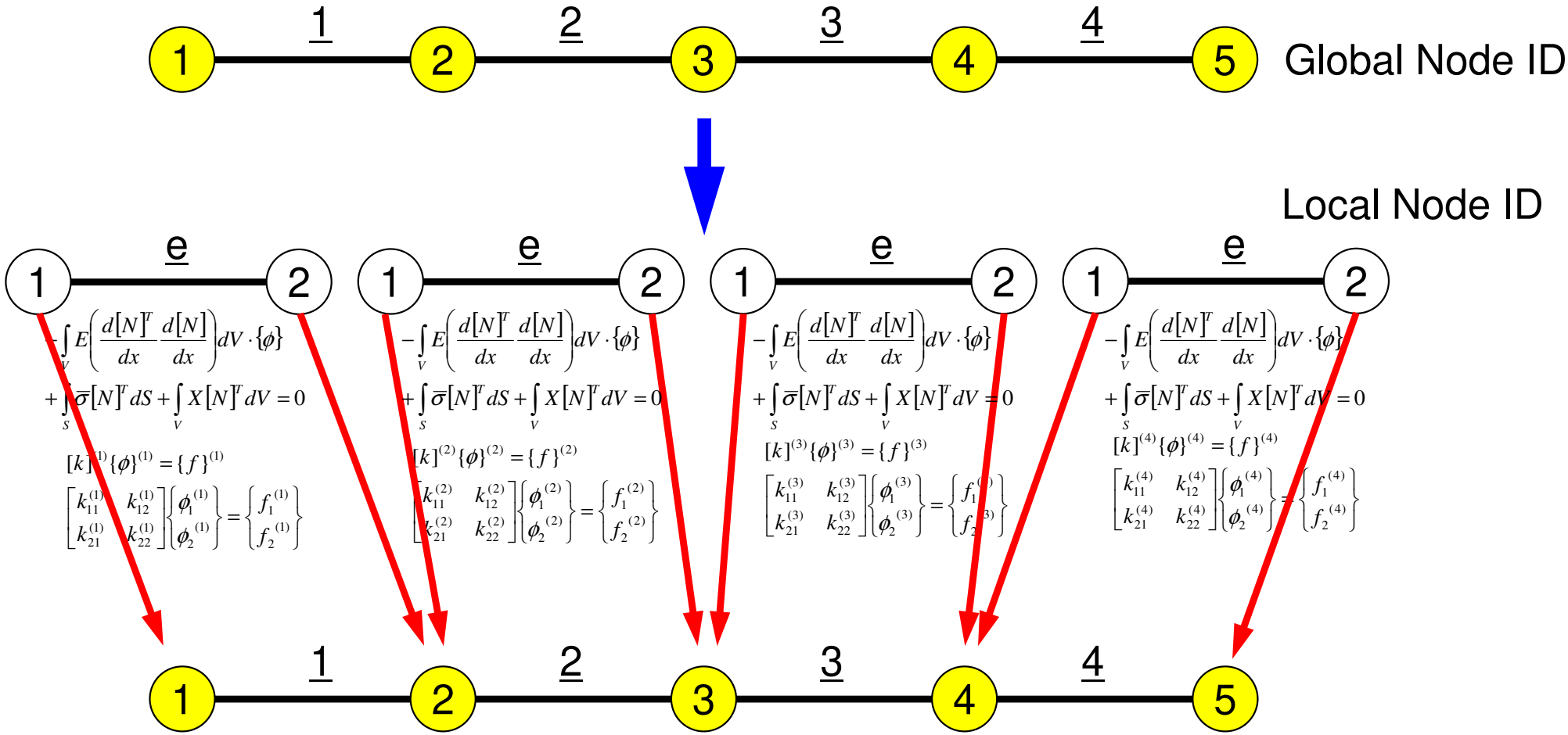
NE: # Element
N : # Node (NE+1)

Program: 1d.c (2/6)
Initialization, Allocation of Arrays

```
/*
// +-------+
// | INIT. |
// +-------+
*/
      fp = fopen("input.dat", "r");
      assert(fp != NULL);
      fscanf(fp, "%d", &NE);
      fscanf(fp, "%lf %lf %lf %lf", &dX, &QV, &Area, &COND);
      fscanf(fp, "%d", &IterMax);
      fscanf(fp, "%lf", &Eps);
      fclose(fp);

      N= NE + 1;

      PHI  = calloc(N, sizeof(double));
      X    = calloc(N, sizeof(double));
      Diag = calloc(N, sizeof(double));

      AMat = calloc(2*N-2, sizeof(double));

      Rhs = calloc(N, sizeof(double));

      Index= calloc(N+1, sizeof(int));
      Item = calloc(2*N-2, sizeof(int));

      Icelnod= calloc(2*NE, sizeof(int));
```

Amat: Non-Zero Off-Diag. Comp.
Item: Corresponding Column ID

Element/Global Operations

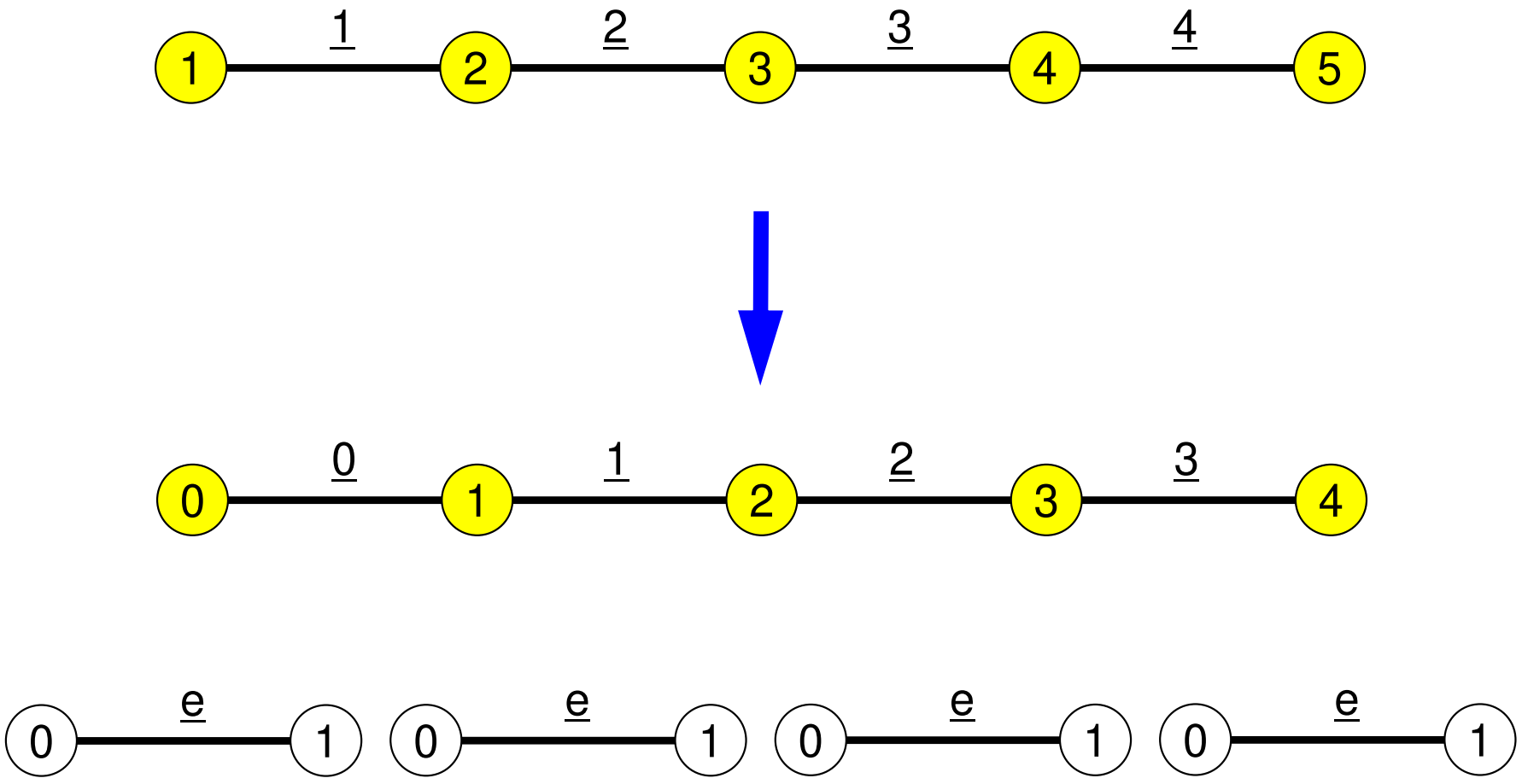


$$[K]\{\Phi\}=\{F\}$$

$$\begin{bmatrix} D_1 & AU_{11} & & & \\ AL_{21} & D_2 & AU_{21} & & \\ & AL_{31} & D_3 & AU_{31} & \\ & & AL_{41} & D_4 & AU_{41} \\ & & & AL_{51} & D_5 \end{bmatrix}\begin{Bmatrix}\Phi_1\\ \Phi_2\\ \Phi_3\\ \Phi_4\\ \Phi_5\end{Bmatrix}=\begin{Bmatrix}B_1\\ B_2\\ B_3\\ B_4\\ B_5\end{Bmatrix}$$

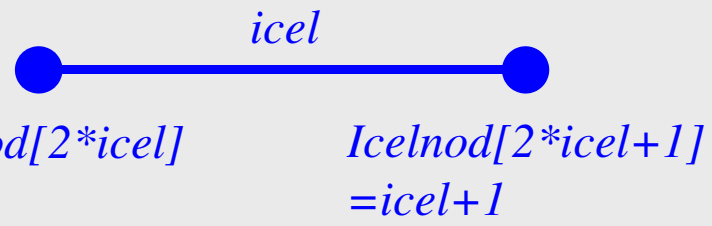
Number of non-zero off-diag. components is 2 for each node. This number is 1 at boundary nodes).

Attention: In C program, node and element ID's start from 0.



Program: 1d.c (2/6)
Initialization, Allocation of Arrays

```
/*
// +-------+
// | INIT. |
// +-------+
*/
      fp = fopen("input.dat", "r");
      assert(fp != NULL);
      fscanf(fp, "%d", &NE);
      fscanf(fp, "%lf %lf %lf %lf", &dX, &QV, &Area, &COND);
      fscanf(fp, "%d", &IterMax);
      fscanf(fp, "%lf", &Eps);
      fclose(fp);

      N= NE + 1;

      PHI  = calloc(N, sizeof(double));
      X    = calloc(N, sizeof(double));
      Diag = calloc(N, sizeof(double));

      AMat = calloc(2*N-2, sizeof(double));

      Rhs = calloc(N, sizeof(double));

      Index= calloc(N+1, sizeof(int));
      Item = calloc(2*N-2, sizeof(int));

      Icelnod= calloc(2*NE, sizeof(int));
```



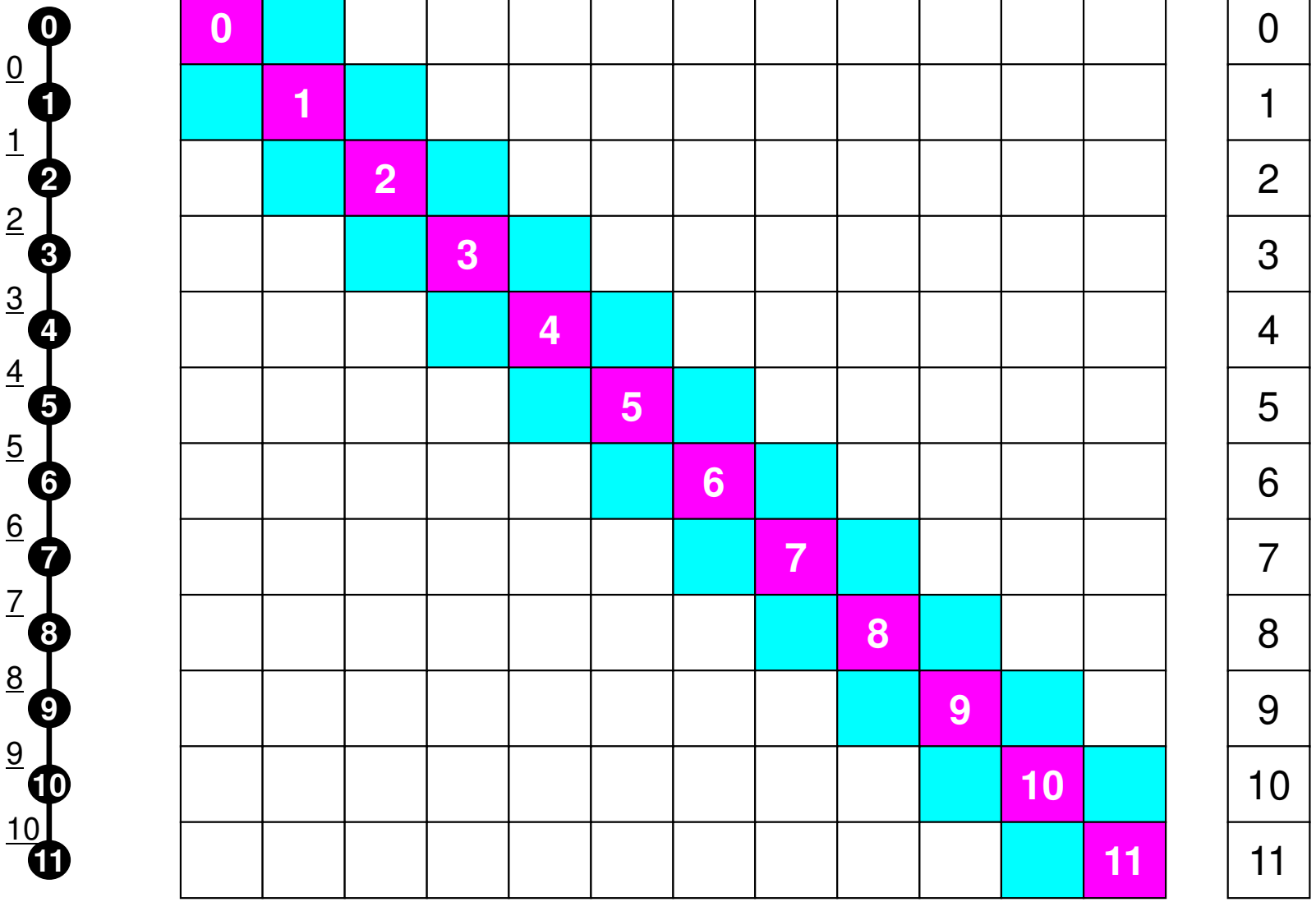
Amat: Non-Zero Off-Diag. Comp.
Item: Corresponding Column ID

Number of non-zero off-diag. components is 2 for each node. This number is 1 at boundary nodes).

Total Number of Non-Zero Off-Diag. Components:
2*(N-2)+1+1= 2*N-2

1D FEM: 12 nodes/11 elem's

三重対角行列：Tri-Diagonal Matrix



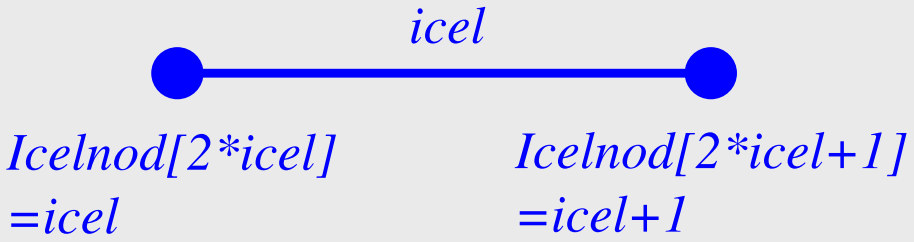
Program: 1d.c (3/6)
Initialization, Allocation of Arrays (cont.)

```
W = (double **)malloc(sizeof(double *)*4);
if(W == NULL) {
  fprintf(stderr, "Error: %s¥n", strerror(errno));
  return -1;
}
for(i=0; i<4; i++) {
  W[i] = (double *)malloc(sizeof(double)*N);
  if(W[i] == NULL) {
    fprintf(stderr, "Error: %s¥n", strerror(errno));
    return -1;
  }
}

for(i=0;i<N;i++)  PHI[i] = 0.0;
for(i=0;i<N;i++) Diag[i] = 0.0;
for(i=0;i<N;i++)  Rhs[i] = 0.0;
for(k=0;k<2*N-2;k++)  AMat[k] = 0.0;

for(i=0;i<N;i++) X[i]= i*dX;

for(icel=0;icel<NE;icel++){
  Icelnod[2*icel  ]= icel;
  Icelnod[2*icel+1]= icel+1;
}

Kmat[0][0]= +1.0;
Kmat[0][1]= -1.0;
Kmat[1][0]= -1.0;
Kmat[1][1]= +1.0;
```

x: X-coordinate component of each node

Program: 1d.c (3/6)
Initialization, Allocation of Arrays (cont.)

```
W = (double **)malloc(sizeof(double *)*4);
      if(W == NULL) {
        fprintf(stderr, "Error: %s¥n", strerror(errno));
        return -1;
      }
      for(i=0; i<4; i++) {
        W[i] = (double *)malloc(sizeof(double)*N);
        if(W[i] == NULL) {
          fprintf(stderr, "Error: %s¥n", strerror(errno));
          return -1;
        }
      }

      for(i=0;i<N;i++)  PHI[i] = 0.0;
      for(i=0;i<N;i++) Diag[i] = 0.0;
      for(i=0;i<N;i++)  Rhs[i] = 0.0;
      for(k=0;k<2*N-2;k++)  AMat[k] = 0.0;

      for(i=0;i<N;i++) X[i]= i*dX;

      for(icel=0;icel<NE;icel++){
        Icelnod[2*icel  ]= icel;
        Icelnod[2*icel+1]= icel+1;
      }

      Kmat[0][0]= +1.0;
      Kmat[0][1]= -1.0;
      Kmat[1][0]= -1.0;
      Kmat[1][1]= +1.0;
```



Program: 1d.c (3/6)

Initialization, Allocation of Arrays (cont.)

```
W = (double **)malloc(sizeof(double *)*4);
if(W == NULL) {
  fprintf(stderr, "Error: %s¥n", strerror(errno));
  return -1;
}
for(i=0; i<4; i++) {
  W[i] = (double *)malloc(sizeof(double)*N);
  if(W[i] == NULL) {
    fprintf(stderr, "Error: %s¥n", strerror(errno));
    return -1;
  }
}

for(i=0;i<N;i++)  PHI[i] = 0.0;
for(i=0;i<N;i++) Diag[i] = 0.0;
for(i=0;i<N;i++)  Rhs[i] = 0.0;
for(k=0;k<2*N-2;k++)  AMat[k] = 0.0;

for(i=0;i<N;i++) X[i]= i*dX;

for(icel=0;icel<NE;icel++){
  Icelnod[2*icel  ]= icel;
  Icelnod[2*icel+1]= icel+1;
}

Kmat[0][0]= +1.0;
Kmat[0][1]= -1.0;
Kmat[1][0]= -1.0;
Kmat[1][1]= +1.0;
```

$$[k]^{(e)} = \int_V \lambda \left(\frac{d[N]^T}{dx} \frac{d[N]}{dx} \right) dV = \frac{\lambda A}{L} \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix}$$

[Kmat]

Program: 1d.c (4/6)

Global Matrix: Column ID for Non-Zero Off-Diag's

```
/*
// +--------------+
// | CONNECTIVITY |
// +--------------+
*/
  for(i=0;i<N+1;i++)  Index[i] = 2;
  Index[0]= 0;
  Index[1]= 1;
  Index[N]= 1;

  for(i=0;i<N;i++){
          Index[i+1]= Index[i+1] + Index[i];
          }

  NPLU= Index[N];

  for(i=0;i<N;i++){
          jS = Index[i];
          if(i == 0){
                  Item[jS] = i+1;
          }else if(i == N-1){
                  Item[jS] = i-1;
          }else{
                  Item[jS] = i-1;
                  Item[jS+1] = i+1;
          }
  }
```

Number of non-zero off-diag. components is 2 for each node. This number is 1 at boundary nodes).

Total Number of Non-Zero Off-Diag. Components:
2*(N-2)+1+1= 2*N-2= NPLU= Index[N]

						# Non-Zero Off-Diag.	
0	1.1 ◎	2.4 ①	3.2 ④			2	Index[0]= 0
1	3.6 ①	4.3 ◎	2.5 ③	3.7 ⑤	9.1 ⑦	4	Index[1]= 2
2	5.7 ②	1.5 ④	3.1 ⑥			2	Index[2]= 6
3	9.8 ③	4.1 ①	2.5 ④	2.7 ⑤		3	Index[3]= 8
4	11.5 ④	3.1 ◎	9.5 ①	10.4 ②	4.3 ⑥	4	Index[4]= 11
5	12.4 ⑤	6.5 ②	9.5 ⑥			2	Index[5]= 15
6	23.1 ⑥	6.4 ①	2.5 ②	1.4 ⑤	13.1 ⑦	4	Index[6]= 17
7	51.3 ⑦	9.5 ①	1.3 ②	9.6 ③	3.1 ⑤	4	Index[7]= 21
							Index[8]= 25

(Index[i])th~(Index[i+1]) th:
Non-Zero Off-Diag. Components corresponding to *i*-th row.

Program: 1d.c (4/6)

Global Matrix: Column ID for Non-Zero Off-Diag's

```
/*
// +--------------+
// | CONNECTIVITY |
// +--------------+
*/
  for(i=0;i<N+1;i++)  Index[i] = 2;
  Index[0]= 0;
  Index[1]= 1;
  Index[N]= 1;

  for(i=0;i<N;i++){
          Index[i+1]= Index[i+1] + Index[i];
          }

  NPLU= Index[N];

  for(i=0;i<N;i++){
          jS = Index[i];
          if(i == 0){
                  Item[jS] = i+1;
          }else if(i == N-1){
                  Item[jS] = i-1;
          }else{
                  Item[jS] = i-1;
                  Item[jS+1] = i+1;
          }
  }
```

						# Non-Zero Off-Diag.	
0	1.1 ◎	2.4 ①	3.2 ④				Index[0]= 0
						2	Index[1]= 2
1	3.6 ①	4.3 ◎	2.5 ③	3.7 ⑤	9.1 ⑦		
						4	Index[2]= 6
2	5.7 ②	1.5 ④	3.1 ⑥				
						2	Index[3]= 8
3	9.8 ③	4.1 ①	2.5 ④	2.7 ⑤			
						3	Index[4]= 11
4	11.5 ④	3.1 ◎	9.5 ①	10.4 ②	4.3 ⑥		
						4	Index[5]= 15
5	12.4 ⑤	6.5 ②	9.5 ⑥				
						2	Index[6]= 17
6	23.1 ⑥	6.4 ①	2.5 ②	1.4 ⑤	13.1 ⑦		
						4	Index[7]= 21
7	51.3 ⑦	9.5 ①	1.3 ②	9.6 ③	3.1 ⑤		
						4	Index[8]= 25

(Index[i])th~(Index[i+1]) th:
Non-Zero Off-Diag. Components corresponding to i-th row.

Program: 1d.c (5/6)

Element Matrix ~ Global Matrix

```
/*
// +----------------+
// | MATRIX assemble |
// +----------------+
*/
      for(icel=0;icel<NE;icel++){
        in1= Icelnod[2*icel];
        in2= Icelnod[2*icel+1];
        X1 = X[in1];
        X2 = X[in2];
        DL = fabs(X2-X1);

        Ck= Area*COND/DL;
        Emat[0][0]= Ck*Kmat[0][0];
        Emat[0][1]= Ck*Kmat[0][1];
        Emat[1][0]= Ck*Kmat[1][0];
        Emat[1][1]= Ck*Kmat[1][1];

        Diag[in1]= Diag[in1] + Emat[0][0];
        Diag[in2]= Diag[in2] + Emat[1][1];

        if (icel==0){k1=Index[in1];
        }else {k1=Index[in1]+1;}
        k2=Index[in2];

        AMat[k1]= AMat[k1] + Emat[0][1];
        AMat[k2]= AMat[k2] + Emat[1][0];

        QN= 0.5*QV*Area*dX;
        Rhs[in1]= Rhs[in1] + QN;
        Rhs[in2]= Rhs[in2] + QN;
      }
```

Program: 1d.c (5/6)

Element Matrix ~ Global Matrix

```
/*
// +----------------+
// | MATRIX assemble |
// +----------------+
*/
      for(icel=0;icel<NE;icel++){
        in1= Icelnod[2*icel];
        in2= Icelnod[2*icel+1];
        X1 = X[in1];
        X2 = X[in2];
        DL = fabs(X2-X1);

        Ck= Area*COND/DL;
        Emat[0][0]= Ck*Kmat[0][0];
        Emat[0][1]= Ck*Kmat[0][1];
        Emat[1][0]= Ck*Kmat[1][0];
        Emat[1][1]= Ck*Kmat[1][1];

        Diag[in1]= Diag[in1] + Emat[0][0];
        Diag[in2]= Diag[in2] + Emat[1][1];

        if (icel==0){k1=Index[in1];
        }else {k1=Index[in1]+1;}
        k2=Index[in2];

        AMat[k1]= AMat[k1] + Emat[0][1];
        AMat[k2]= AMat[k2] + Emat[1][0];

        QN= 0.5*QV*Area*dX;
        Rhs[in1]= Rhs[in1] + QN;
        Rhs[in2]= Rhs[in2] + QN;
      }
```

$$[Emat]=[k]^{(e)}=\frac{\lambda A}{L}\begin{bmatrix}+1 & -1\\ -1 & +1\end{bmatrix}=\frac{\lambda A}{L}[Kmat]$$

Program: 1d.c (5/6)
Element Matrix ~ Global Matrix

```
/*
// +----------------+
// | MATRIX assemble |
// +----------------+
*/
      for(icel=0;icel<NE;icel++){
        in1= Icelnod[2*icel];
        in2= Icelnod[2*icel+1];
        X1 = X[in1];
        X2 = X[in2];
        DL = fabs(X2-X1);

        Ck= Area*COND/DL;
        Emat[0][0]= Ck*Kmat[0][0];
        Emat[0][1]= Ck*Kmat[0][1];
        Emat[1][0]= Ck*Kmat[1][0];
        Emat[1][1]= Ck*Kmat[1][1];

        Diag[in1]= Diag[in1] + Emat[0][0];
        Diag[in2]= Diag[in2] + Emat[1][1];

        if (icel==0){k1=Index[in1];
        }else {k1=Index[in1]+1;}
        k2=Index[in2];

        AMat[k1]= AMat[k1] + Emat[0][1];
        AMat[k2]= AMat[k2] + Emat[1][0];

        QN= 0.5*QV*Area*dX;
        Rhs[in1]= Rhs[in1] + QN;
        Rhs[in2]= Rhs[in2] + QN;
      }
```

$$[Emat]=[k]^{(e)}=\frac{\lambda A}{L}\begin{bmatrix}+1 & -1\\ -1 & +1\end{bmatrix}$$

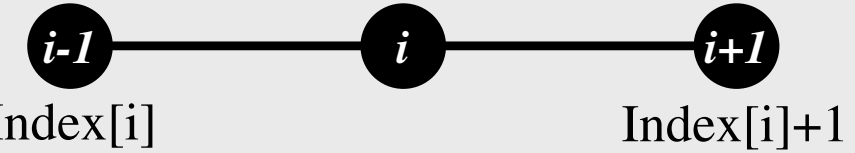
Program: 1d.c (5/6)

Element Matrix ~ Global Matrix

```
/*
// +-----------------+
// | MATRIX assemble |
// +-----------------+
*/
      for(icel=0;icel<NE;icel++){
        in1= Icelnod[2*icel];
        in2= Icelnod[2*icel+1];
        X1 = X[in1];
        X2 = X[in2];
        DL = fabs(X2-X1);

        Ck= Area*COND/DL;
        Emat[0][0]= Ck*Kmat[0][0];
        Emat[0][1]= Ck*Kmat[0][1];
        Emat[1][0]= Ck*Kmat[1][0];
        Emat[1][1]= Ck*Kmat[1][1];

        Diag[in1]= Diag[in1] + Emat[0][0];
        Diag[in2]= Diag[in2] + Emat[1][1];

        if (icel==0){k1=Index[in1];
        }else {k1=Index[in1]+1;}
        k2=Index[in2];

        AMat[k1]= AMat[k1] + Emat[0][1];
        AMat[k2]= AMat[k2] + Emat[1][0];

        QN= 0.5*QV*Area*dX;
        Rhs[in1]= Rhs[in1] + QN;
        Rhs[in2]= Rhs[in2] + QN;
      }
```

Non-zero Off-Diag. at i-th row:

`Index[i], Index[i]+1`



$$[Emat] = [k]^{(e)} = \frac{\lambda A}{L}\begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix}$$

k1

k2

General Elements: k1

"in2" as a off-diag. component of "in1"

```
/*
// +-----------------+
// | MATRIX assemble |
// +-----------------+
*/
      for(icel=0;icel<NE;icel++){
        in1= Icelnod[2*icel];
        in2= Icelnod[2*icel+1];
        X1 = X[in1];
        X2 = X[in2];
        DL = fabs(X2-X1);

        Ck= Area*COND/DL;
        Emat[0][0]= Ck*Kmat[0][0];
        Emat[0][1]= Ck*Kmat[0][1];
        Emat[1][0]= Ck*Kmat[1][0];
        Emat[1][1]= Ck*Kmat[1][1];

        Diag[in1]= Diag[in1] + Emat[0][0];
        Diag[in2]= Diag[in2] + Emat[1][1];

        if (icel==0){k1=Index[in1];
        }else {k1=Index[in1]+1;}
        k2=Index[in2];

        AMat[k1]= AMat[k1] + Emat[0][1];
        AMat[k2]= AMat[k2] + Emat[1][0];

        QN= 0.5*QV*Area*dX;
        Rhs[in1]= Rhs[in1] + QN;
        Rhs[in2]= Rhs[in2] + QN;
      }
```

Non-zero Off-Diag. at i-th row:
`Index[i], Index[i]+1`

$$[Emat]=[k]^{(e)}=\frac{\lambda A}{L}\begin{bmatrix}+1 & -1\\ -1 & +1\end{bmatrix}$$

k1

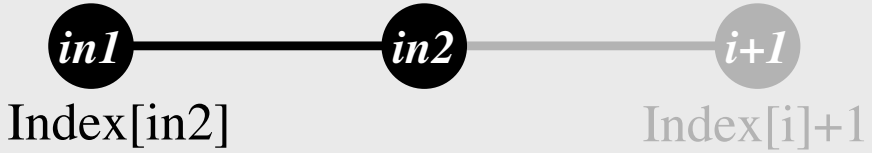
General Elements: k2

"in1" as a off-diag. component of "in2"

```
/*
// +-----------------+
// | MATRIX assemble |
// +-----------------+
*/
      for(icel=0;icel<NE;icel++){
        in1= Icelnod[2*icel];
        in2= Icelnod[2*icel+1];
        X1 = X[in1];
        X2 = X[in2];
        DL = fabs(X2-X1);

        Ck= Area*COND/DL;
        Emat[0][0]= Ck*Kmat[0][0];
        Emat[0][1]= Ck*Kmat[0][1];
        Emat[1][0]= Ck*Kmat[1][0];
        Emat[1][1]= Ck*Kmat[1][1];

        Diag[in1]= Diag[in1] + Emat[0][0];
        Diag[in2]= Diag[in2] + Emat[1][1];

        if (icel==0){k1=Index[in1];
        }else {k1=Index[in1]+1;}
        k2=Index[in2];

        AMat[k1]= AMat[k1] + Emat[0][1];
        AMat[k2]= AMat[k2] + Emat[1][0];

        QN= 0.5*QV*Area*dX;
        Rhs[in1]= Rhs[in1] + QN;
        Rhs[in2]= Rhs[in2] + QN;
      }
```

Non-zero Off-Diag. at i-th row:

`Index[i], Index[i]+1`



$$[Emat] = [k]^{(e)} = \frac{\lambda A}{L}\begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix}$$

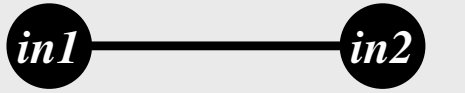
k2

0-th Element: k1

"in2" as a off-diag. component of "in1"

```
/*
// +-----------------+
// | MATRIX assemble |
// +-----------------+
*/
      for(icel=0;icel<NE;icel++){
        in1= Icelnod[2*icel];
        in2= Icelnod[2*icel+1];
        X1 = X[in1];
        X2 = X[in2];
        DL = fabs(X2-X1);

        Ck= Area*COND/DL;
        Emat[0][0]= Ck*Kmat[0][0];
        Emat[0][1]= Ck*Kmat[0][1];
        Emat[1][0]= Ck*Kmat[1][0];
        Emat[1][1]= Ck*Kmat[1][1];

        Diag[in1]= Diag[in1] + Emat[0][0];
        Diag[in2]= Diag[in2] + Emat[1][1];

        if (icel==0){k1=Index[in1];
        }else {k1=Index[in1]+1;}
        k2=Index[in2];

        AMat[k1]= AMat[k1] + Emat[0][1];
        AMat[k2]= AMat[k2] + Emat[1][0];

        QN= 0.5*QV*Area*dX;
        Rhs[in1]= Rhs[in1] + QN;
        Rhs[in2]= Rhs[in2] + QN;
      }
```

Non-zero Off-Diag. at i-th row: **`Index[i]`**



Index[in1]

$$[Emat]=[k]^{(e)}=\frac{\lambda A}{L}\begin{bmatrix}+1 & -1\\ -1 & +1\end{bmatrix}$$

k1

Program: 1d.c (5/6)
RHS: Heat Generation Term

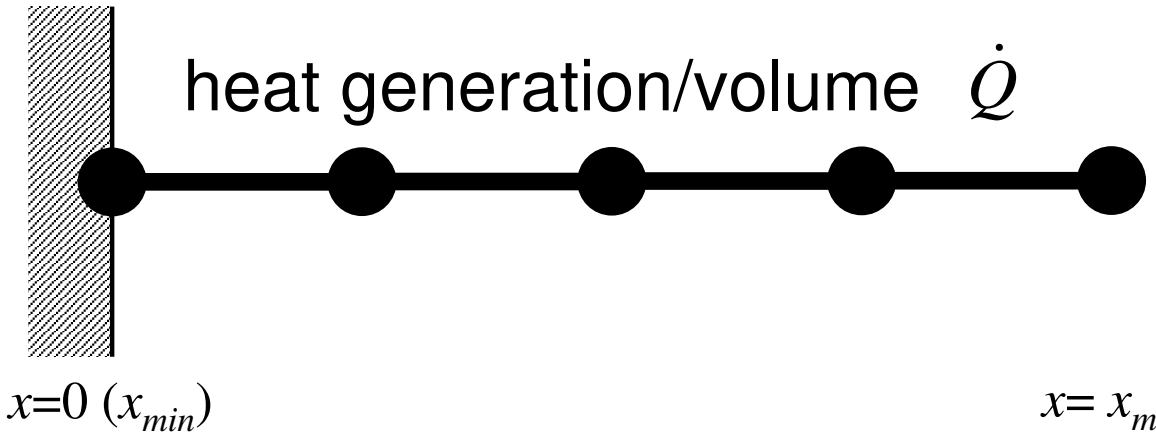
```
/*
// +----------------+
// | MATRIX assemble |
// +----------------+
*/
      for(icel=0;icel<NE;icel++){
        in1= Icelnod[2*icel];
        in2= Icelnod[2*icel+1];
        X1 = X[in1];
        X2 = X[in2];
        DL = fabs(X2-X1);

        Ck= Area*COND/DL;
        Emat[0][0]= Ck*Kmat[0][0];
        Emat[0][1]= Ck*Kmat[0][1];
        Emat[1][0]= Ck*Kmat[1][0];
        Emat[1][1]= Ck*Kmat[1][1];

        Diag[in1]= Diag[in1] + Emat[0][0];
        Diag[in2]= Diag[in2] + Emat[1][1];

        if (icel==0){k1=Index[in1];
        }else {k1=Index[in1]+1;}
        k2=Index[in2];

        AMat[k1]= AMat[k1] + Emat[0][1];
        AMat[k2]= AMat[k2] + Emat[1][0];

        QN= 0.5*QV*Area*dX;
        Rhs[in1]= Rhs[in1] + QN;
        Rhs[in2]= Rhs[in2] + QN;
      }
```

$$\int_V \dot{Q}[N]^T dV = \dot{Q}A\int_0^L \begin{bmatrix} 1-x/L \\ x/L \end{bmatrix} dx = \frac{\dot{Q}AL}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Program: 1d.c (6/6)
Dirichlet B.C. @ X=0

```
/*
// +--------------------+
// | BOUNDARY conditions |
// +--------------------+
*/

/* X=Xmin */
        i=0;
        jS= Index[i];
        AMat[jS]= 0.0;
        Diag[i ]= 1.0;
        Rhs [i ]= 0.0;

        for(k=0;k<NPLU;k++){
          if(Item[k]==0){AMat[k]=0.0;
          }}
```

1D Steady State Heat Conduction

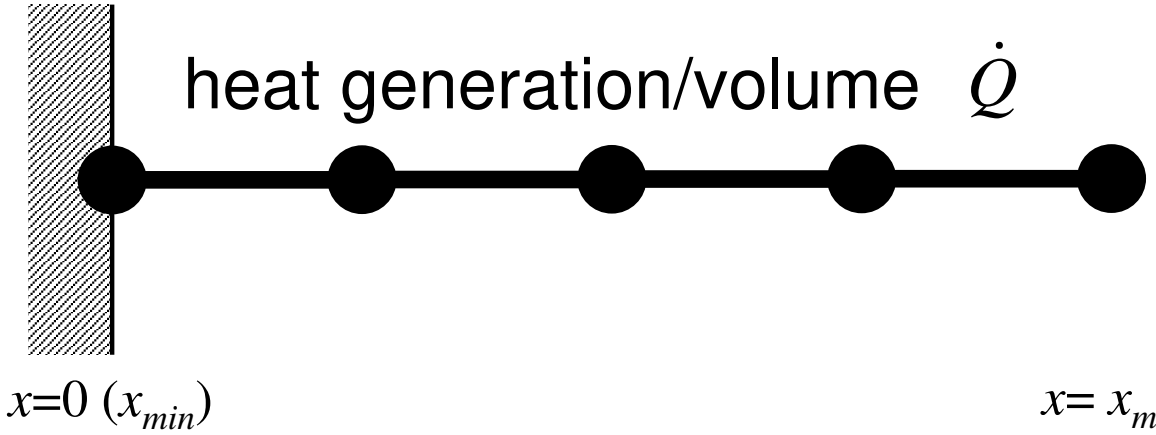


$$\frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right)+\dot{Q}=0$$

- Uniform: Sectional Area: A, Thermal Conductivity: λ
- Heat Generation Rate/Volume/Time [$QL^{-3}T^{-1}$] $\dot{Q}$
- Boundary Conditions
  - x=0 : T= 0 (Fixed Temperature)
  - x=x_{max} : $\dfrac{\partial T}{\partial x}=0$ (Insulated)

(Linear) Equation at x=0

$T_1 = 0$ (or $T_0 = 0$)



$$\frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right)+\dot{Q}=0$$

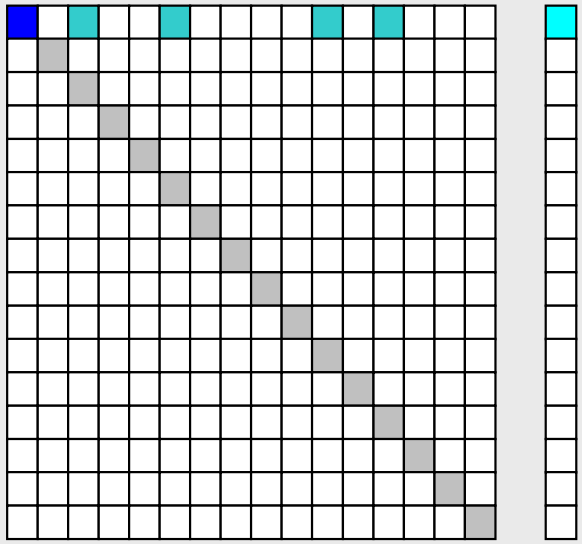
- Uniform: Sectional Area: A, Thermal Conductivity: λ
- Heat Generation Rate/Volume/Time [$QL^{-3}T^{-1}$] $\dot{Q}$
- Boundary Conditions
  - x=0 : T= 0 (Fixed Temperature)
  - x=x_{max} : $\dfrac{\partial T}{\partial x}=0$ (Insulated)

Program: 1d.c (6/6)
Dirichlet B.C. @ X=0

```
/*
// +--------------------+
// | BOUNDARY conditions |
// +--------------------+
*/

/* X=Xmin */
        i=0;
        jS= Index[i];
        AMat[jS]= 0.0;
        Diag[i ]= 1.0;
        Rhs [i ]= 0.0;

        for(k=0;k<NPLU;k++){
          if(Item[k]==0){AMat[k]=0.0;
          }}
```

$T_1=0$

Diagonal Component=1
RHS=0
Off-Diagonal Components= 0.



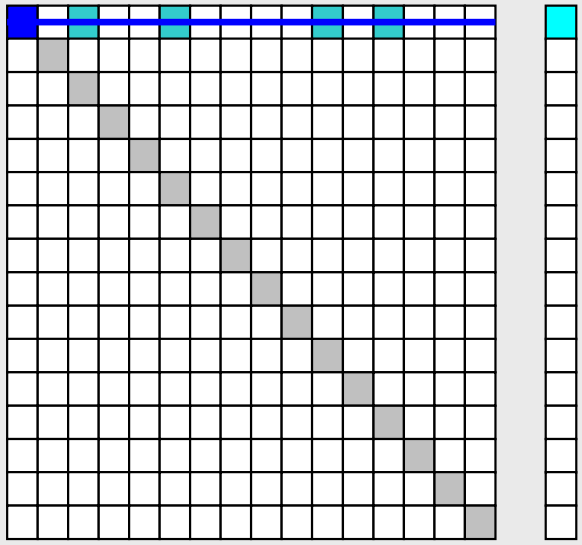
Program: 1d.c (6/6)
Dirichlet B.C. @ X=0

```
/*
// +--------------------+
// | BOUNDARY conditions |
// +--------------------+
*/

/* X=Xmin */
        i=0;
        jS= Index[i];
        AMat[jS]= 0.0;
        Diag[i ]= 1.0;
        Rhs [i ]= 0.0;

        for(k=0;k<NPLU;k++){
          if(Item[k]==0){AMat[k]=0.0;
          }}
```

$T_1=0$

Diagonal Component=1
RHS=0
Off-Diagonal Components= 0.

Erase !



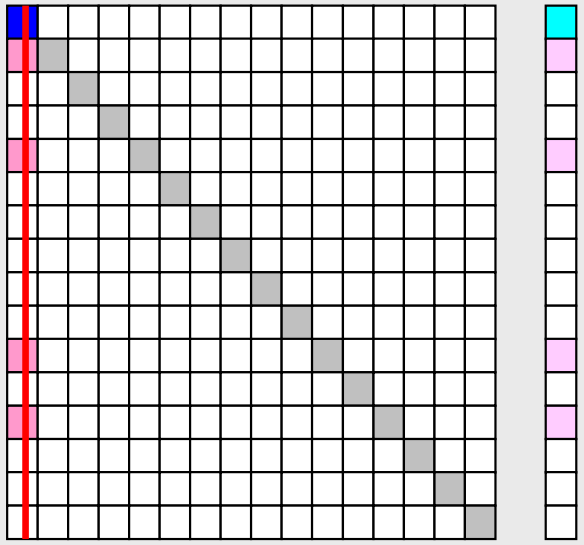
Program: 1d.c (6/6)
Dirichlet B.C. @ X=0

```
/*
// +--------------------+
// | BOUNDARY conditions |
// +--------------------+
*/

/* X=Xmin */
        i=0;
        jS= Index[i];
        AMat[jS]= 0.0;
        Diag[i ]= 1.0;
        Rhs [i ]= 0.0;

        for(k=0;k<NPLU;k++){
          if(Item[k]==0){AMat[k]=0.0;
          }}
```

$T_1=0$

Diagonal Component=1
RHS=0
Off-Diagonal Components= 0.

Elimination and Erase



Column components of boundary nodes (Dirichlet B.C.) are moved to RHS and eliminated for keeping symmetrical feature of the matrix (in this case just erase off-diagonal components)

if $T_1 \neq 0$

```
/*
// +--------------------+
// | BOUNDARY conditions |
// +--------------------+
*/
```

Column components of boundary nodes (Dirichlet B.C.) are moved to RHS and eliminated for keeping symmetrical feature of the matrix.

$$Diag_j \phi_j + \sum_{k=Index[j]}^{Index[j+1]-1} Amat_k \ \phi_{Item[k]} = Rhs_j$$

```
/* X=Xmin */
        i=0;
        jS= Index[i];
        AMat[jS]= 0.0;
        Diag[i ]= 1.0;
        Rhs [i ]= PHImin;

          for(j=1;i<N;i++){
            for (k=Index[j];k<Index[j+1];k++){
              if(Item[k]==0){
                 Rhs [j]= Rhs[j] - Amat[k]*PHImin;
                 AMat[k]= 0.0;
            }}
```

if $T_I \neq 0$

```
/*
// +--------------------+
// | BOUNDARY conditions |
// +--------------------+
*/



/* X=Xmin */
        i=0;
        jS= Index[i];
        AMat[jS]= 0.0;
        Diag[i ]= 1.0;
        Rhs [i ]= PHImin;

          for(j=1;i<N;i++){
            for (k=Index[j];k<Index[j+1];k++){
              if(Item[k]==0){
                 Rhs [j]= Rhs[j] - Amat[k]*PHImin;
                 AMat[k]= 0.0;
            }}
```

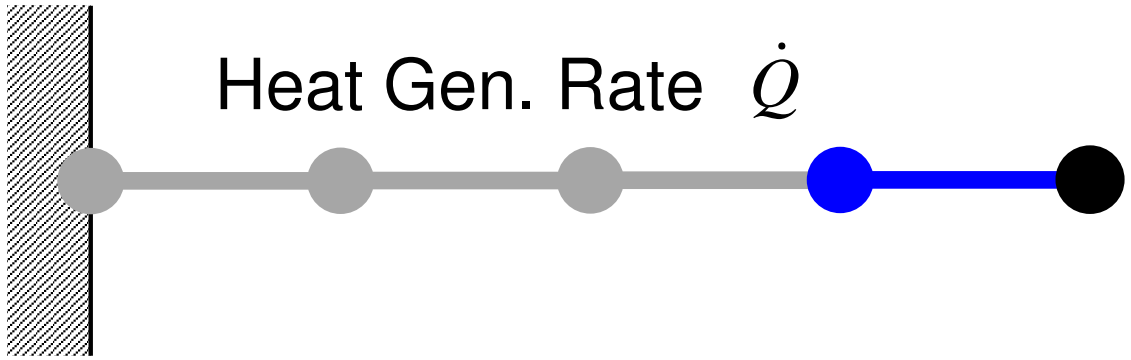
$$Diag_j \phi_j + \sum_{k=Index[j],k\neq k_s}^{Index[j+1]-1} Amat_k \ \phi_{Item[k]}$$

$$= Rhs_j - Amat_{k_s} \phi_{Item[k_s]}$$

$$= Rhs_j - Amat_{k_s} \phi_{\min} \quad where\ Item[k_s] = 0$$

Column components of boundary nodes (Dirichlet B.C.) are moved to RHS and eliminated for keeping symmetrical feature of the matrix.

Secondary B.C. (Insulated)



$$\frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right)+\dot{Q}=0$$

$$T=0 \,@\, x=0$$

$$\frac{\partial T}{\partial x}=0 \,@\, x=x_{\max}$$

$$\int_S \bar{q}[N]^T dS=\bar{q}A\big|_{x=L}=\bar{q}A\begin{Bmatrix}0\\1\end{Bmatrix},\quad \bar{q}=-\lambda\frac{dT}{dx}$$

Surface Flux

$$\frac{\partial T}{\partial x}=0 \,@\, x=x_{\max}$$

According to insulated B.C., $\bar{q}=0$
is satisfied. No contribution by this term.
Insulated B.C. is automatically satisfied
without explicit operations
-> Natural B.C.

Preconditioned CG Solver

Compute $\mathbf{r}^{(0)} = \mathbf{b} - [\mathbf{A}]\mathbf{x}^{(0)}$
for i= 1, 2, …
  solve $[\mathbf{M}]\mathbf{z}^{(i-1)} = \mathbf{r}^{(i-1)}$
  $\rho_{i-1} = \mathbf{r}^{(i-1)}\ \mathbf{z}^{(i-1)}$
  if i=1
    $p^{(1)} = z^{(0)}$
  else
    $\beta_{i-1} = \rho_{i-1}/\rho_{i-2}$
    $p^{(i)} = z^{(i-1)} + \beta_{i-1}\ p^{(i-1)}$
  endif
  $\mathbf{q}^{(i)} = [\mathbf{A}]\mathbf{p}^{(i)}$
  $\alpha_i = \rho_{i-1}/\mathbf{p}^{(i)}\mathbf{q}^{(i)}$
  $x^{(i)} = x^{(i-1)} + \alpha_i p^{(i)}$
  $r^{(i)} = r^{(i-1)} - \alpha_i q^{(i)}$
  check convergence $|\mathbf{r}|$
end

$$[M] = \begin{bmatrix} D_1 & 0 & \dots & 0 & 0 \\ 0 & D_2 & & 0 & 0 \\ \dots & & \dots & & \dots \\ 0 & 0 & & D_{N-1} & 0 \\ 0 & 0 & \dots & 0 & D_N \end{bmatrix}$$

Diagonal Scaling, Point-Jacobi

$$[M] = \begin{bmatrix} D_1 & 0 & \dots & 0 & 0 \\ 0 & D_2 & & 0 & 0 \\ \dots & & \dots & & \dots \\ 0 & 0 & & D_{N-1} & 0 \\ 0 & 0 & \dots & 0 & D_N \end{bmatrix}$$

- **`solve [M]z(i-1)= r(i-1)`** is very easy.
- Provides fast convergence for simple problems.
- 1d.f, 1d.c

CG Solver (1/6)

```
/*
// +---------------+
// | CG iterations |
// +---------------+
*/
        R = 0;
        Z = 1;
        Q = 1;
        P = 2;
        DD= 3;

        for(i=0;i<N;i++){
                W[DD][i]= 1.0 / Diag[i];
        }
```

```
W[0][i]= W[R][i]  ⇒ {r}
W[1][i]= W[Z][i]  ⇒ {z}
W[1][i]= W[Q][i]  ⇒ {q}
W[2][i]= W[P][i]  ⇒ {p}
W[3][i]= W[DD][i] ⇒ 1/{D}
```

Compute $r^{(0)}= b-[A]x^{(0)}$
for i= 1, 2, …
  solve $[M]z^{(i-1)}= r^{(i-1)}$
  $\rho_{i-1}= r^{(i-1)}\ z^{(i-1)}$
  if i=1
    $p^{(1)}= z^{(0)}$
  else
    $\beta_{i-1}= \rho_{i-1}/\rho_{i-2}$
    $p^{(i)}= z^{(i-1)} + \beta_{i-1}\ p^{(i-1)}$
  endif
  $q^{(i)}= [A]p^{(i)}$
  $\alpha_i = \rho_{i-1}/p^{(i)}q^{(i)}$
  $x^{(i)}= x^{(i-1)} + \alpha_i p^{(i)}$
  $r^{(i)}= r^{(i-1)} - \alpha_i q^{(i)}$
  check convergence $|r|$
end

CG Solver (1/6)

```
/*
// +---------------+
// | CG iterations |
// +---------------+
*/
          R = 0;
          Z = 1;
          Q = 1;
          P = 2;
          DD= 3;

          for(i=0;i<N;i++){
                    W[DD][i]= 1.0 / Diag[i];
          }
```

Reciprocal numbers（逆数）of diagonal components are stored in `W[DD][i]`.
Computational cost for division is usually expensive.

Although it was said (division):(+,-,*) is 10:1 before, the difference is much smaller now. Generally, multiplying is still faster than division.

$$
\begin{aligned}
W[0][i] &= W[R][i] \Rightarrow \{r\} \\
W[1][i] &= W[Z][i] \Rightarrow \{z\} \\
W[1][i] &= W[Q][i] \Rightarrow \{q\} \\
W[2][i] &= W[P][i] \Rightarrow \{p\} \\
W[3][i] &= W[DD][i] \Rightarrow 1/\{D\}
\end{aligned}
$$

CG Solver (2/6)

```
/*
//-- {r0}= {b} - [A]{xini} |
*/
        for(i=0;i<N;i++){
                W[R][i] = Diag[i]*PHI[i];
                for(j=Index[i];j<Index[i+1];j++){
                        W[R][i] += AMat[j]*PHI[Item[j]];
                }
        }

        BNorm2 = 0.0;
        for(i=0;i<N;i++){
                BNorm2 += Rhs[i] * Rhs[i];
                W[R][i] = Rhs[i] - W[R][i];
        }
```

BNRM2=$|b|^2$
for convergence criteria of CG solvers

Compute $r^{(0)}= b-[A]x^{(0)}$

for i= 1, 2, …
  solve $[M]z^{(i-1)}= r^{(i-1)}$
  $\rho_{i-1}= r^{(i-1)}\ z^{(i-1)}$
  if i=1
    $p^{(1)}= z^{(0)}$
  else
    $\beta_{i-1}= \rho_{i-1}/\rho_{i-2}$
    $p^{(i)}= z^{(i-1)} + \beta_{i-1}\ p^{(i-1)}$
  endif
  $q^{(i)}= [A]p^{(i)}$
  $\alpha_i = \rho_{i-1}/p^{(i)}q^{(i)}$
  $x^{(i)}= x^{(i-1)} + \alpha_i p^{(i)}$
  $r^{(i)}= r^{(i-1)} - \alpha_i q^{(i)}$
  check convergence $|r|$
end

CG Solver (3/6)

```
 for(iter=1;iter<=IterMax;iter++){

/*
//-- {z}= [Minv]{r}
*/
    for(i=0;i<N;i++){
        W[Z][i] = W[DD][i] * W[R][i];
    }

/*
//-- RHO= {r}{z}
*/
    Rho= 0.0;
    for(i=0;i<N;i++){
        Rho += W[R][i] * W[Z][i];
    }
```

Compute $r^{(0)}= b-[A]x^{(0)}$

for i= 1, 2, …

  solve $[M]z^{(i-1)}= r^{(i-1)}$

  $\rho_{i-1}= r^{(i-1)}\ z^{(i-1)}$

  if i=1

    $p^{(1)}= z^{(0)}$

  else

    $\beta_{i-1}= \rho_{i-1}/\rho_{i-2}$

    $p^{(i)}= z^{(i-1)} + \beta_{i-1}\ p^{(i-1)}$

  endif

  $q^{(i)}= [A]p^{(i)}$

  $\alpha_i = \rho_{i-1}/p^{(i)}q^{(i)}$

  $x^{(i)}= x^{(i-1)} + \alpha_i p^{(i)}$

  $r^{(i)}= r^{(i-1)} - \alpha_i q^{(i)}$

  check convergence $|r|$

end

CG Solver (4/6)

```
/*
//-- {p} = {z} if      ITER=1
//   BETA= RHO / RHO1  otherwise
*/
  if(iter == 1){
     for(i=0;i<N;i++){
         W[P][i] = W[Z][i];
     }
  }else{
         Beta = Rho / Rho1;
         for(i=0;i<N;i++){
             W[P][i] = W[Z][i] + Beta*W[P][i];
         }
  }

/*
//-- {q}= [A]{p}
*/
  for(i=0;i<N;i++){
      W[Q][i] = Diag[i] * W[P][i];
      for(j=Index[i];j<Index[i+1];j++){
          W[Q][i] += AMat[j]*W[P][Item[j]];
          }
      }
```

Compute $r^{(0)}= b-[A]x^{(0)}$

for i= 1, 2, …

- solve $[M]z^{(i-1)}= r^{(i-1)}$
- $\rho_{i-1}= r^{(i-1)}\ z^{(i-1)}$
- **if i=1**
  - **$p^{(1)}= z^{(0)}$**
- **else**
  - **$\beta_{i-1}= \rho_{i-1}/\rho_{i-2}$**
  - **$p^{(i)}= z^{(i-1)} + \beta_{i-1}\ p^{(i-1)}$**
- **endif**
- **$q^{(i)}= [A]p^{(i)}$**
- $\alpha_i = \rho_{i-1}/p^{(i)}q^{(i)}$
- $x^{(i)}= x^{(i-1)} + \alpha_i p^{(i)}$
- $r^{(i)}= r^{(i-1)} - \alpha_i q^{(i)}$
- check convergence $|r|$

end

CG Solver (5/6)

```
/*
//-- ALPHA= RHO / {p}{q}
*/
  C1 = 0.0;
  for(i=0;i<N;i++){
      C1 += W[P][i] * W[Q][i];
  }

  Alpha = Rho / C1;

/*
//-- {x}= {x} + ALPHA*{p}
//   {r}= {r} - ALPHA*{q}
*/
  for(i=0;i<N;i++){
      PHI [i] += Alpha * W[P][i];
      W[R][i] -= Alpha * W[Q][i];
  }
```

Compute $r^{(0)}= b-[A]x^{(0)}$

for i= 1, 2, …

  solve $[M]z^{(i-1)}= r^{(i-1)}$

  $\rho_{i-1}= r^{(i-1)}\ z^{(i-1)}$

  if i=1

    $p^{(1)}= z^{(0)}$

  else

    $\beta_{i-1}= \rho_{i-1}/\rho_{i-2}$

    $p^{(i)}= z^{(i-1)} + \beta_{i-1}\ p^{(i-1)}$

  endif

  $q^{(i)}= [A]p^{(i)}$

  $\alpha_i = \rho_{i-1}/p^{(i)}q^{(i)}$

  $x^{(i)}= x^{(i-1)} + \alpha_i p^{(i)}$

  $r^{(i)}= r^{(i-1)} - \alpha_i q^{(i)}$

  check convergence $|r|$

end

CG Solver (6/6)

```
  DNorm2 = 0.0;
  for(i=0;i<N;i++){
      DNorm2 += W[R][i] * W[R][i];
  }
  Resid = sqrt(DNorm2/BNorm2);

  if((iter)%1000 == 0){
      printf("%8d%s%16.6e¥n", iter, "
             iters, RESID=", Resid);
   }
   if(Resid <= Eps){ierr = 0; break;}
   Rho1 = Rho;   ρ_{i-2}
}
```

Compute $r^{(0)}= b-[A]x^{(0)}$
for i= 1, 2, …
  solve $[M]z^{(i-1)}= r^{(i-1)}$
  $\rho_{i-1}= r^{(i-1)}\ z^{(i-1)}$
  if i=1
    $p^{(1)}= z^{(0)}$
  else
    $\beta_{i-1}= \rho_{i-1}/\rho_{i-2}$
    $p^{(i)}= z^{(i-1)} + \beta_{i-1}\ p^{(i-1)}$
  endif
  $q^{(i)}= [A]p^{(i)}$
  $\alpha_i = \rho_{i-1}/p^{(i)}q^{(i)}$
  $x^{(i)}= x^{(i-1)} + \alpha_i p^{(i)}$
  $r^{(i)}= r^{(i-1)} - \alpha_i q^{(i)}$
  check convergence |r|
end

$$\text{Resid} = \sqrt{\frac{\text{DNorm2}}{\text{BNorm2}}} = \frac{|r|}{|b|} = \frac{|b-Ax|}{|b|} \le \text{Eps}$$

$$|r|, |b| : 2 / L2 / Euclidean - norm \quad (\|r\|_2, \|b\|_2)$$

Control Data **input.dat**

```
4                    NE (Number of Elements)
1.0 1.0 1.0 1.0 Δx (Length of Each Elem.: L), Q, A, λ
100                  Number of MAX. Iterations for CG Solver
1.e-8                Convergence Criteria for CG Solver
```

$$Ax = b \Rightarrow \alpha Ax = \alpha b$$

$$r = b - Ax \Rightarrow R = \alpha b - \alpha Ax = \alpha r$$

Finite Element Procedures

- Initialization
  - Control Data
  - Node, Connectivity of Elements (N: Node#, NE: Elem#)
  - Initialization of Arrays (Global/Element Matrices)
  - Element-Global Matrix Mapping (Index, Item)
- Generation of Matrix
  - Element-by-Element Operations (do icel= 1, NE)
    - Element matrices
    - Accumulation to global matrix
  - Boundary Conditions
- Linear Solver
  - Conjugate Gradient Method

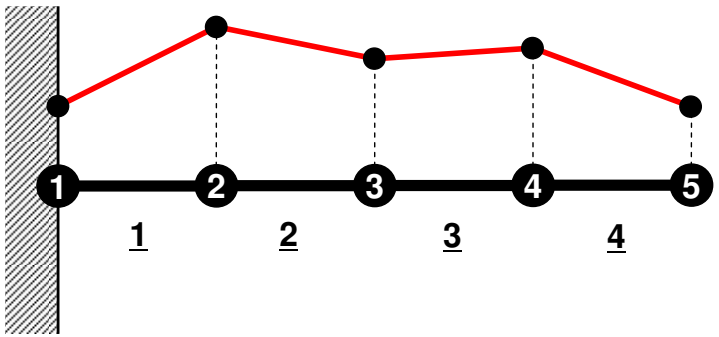
Remedies for Higher Accuracy

- Finer Meshes

```
NE=8, dX=12.5

8 iters, RESID=    2.822910E-16 PHI(N)=     1.953586E-01

### Temperature
       1    0.000000E+00   -0.000000E+00
       2    1.101928E-02    1.103160E-02
       3    2.348034E-02    2.351048E-02
       4    3.781726E-02    3.787457E-02
       5    5.469490E-02    5.479659E-02
       6    7.520772E-02    7.538926E-02
       7    1.013515E-01    1.016991E-01
       8    1.373875E-01    1.381746E-01
       9    1.953586E-01    1.980421E-01
```

$$A = A_1 x + A_2 \quad (>0)$$

```
NE=20, dX=5

 20 iters, RESID=    5.707508E-15 PHI(N)=   1.975734E-01

### Temperature
       1    0.000000E+00   -0.000000E+00
       2    4.259851E-03    4.260561E-03
       3    8.719160E-03    8.720685E-03
       4    1.339752E-02    1.339999E-02
……
      17    1.145876E-01    1.146641E-01
      18    1.295689E-01    1.296764E-01
      19    1.473466E-01    1.475060E-01
      20    1.692046E-01    1.694607E-01
      21    1.975734E-01    1.980421E-01
```



$$u = \frac{F}{EA_1}\left[\log\left(A_1 x + A_2\right) - \log\left(A_2\right)\right]$$

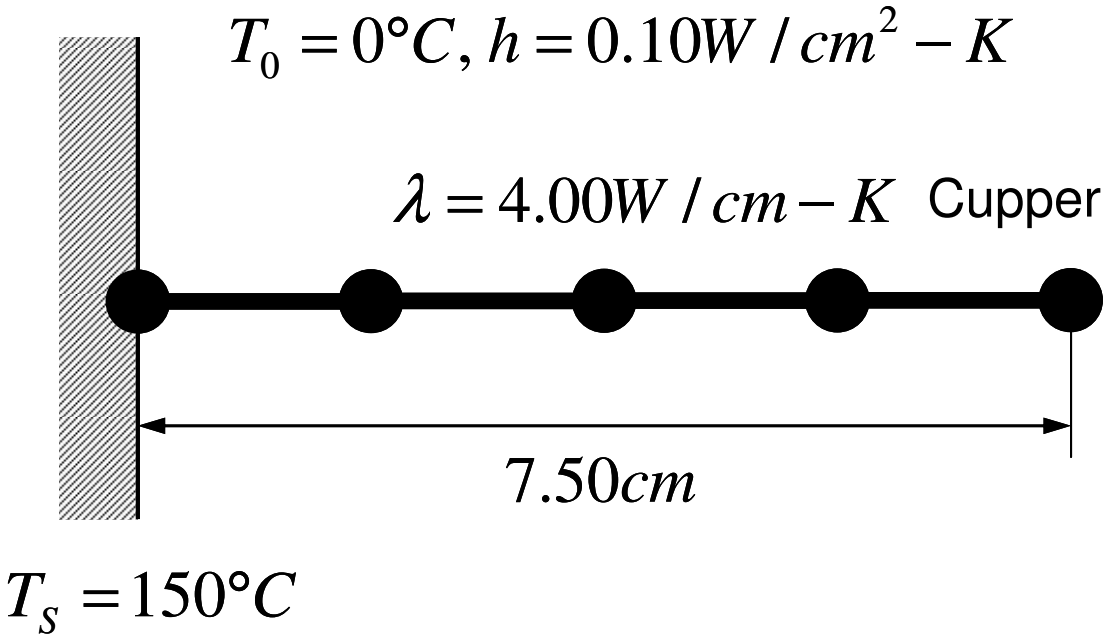
Remedies for Higher Accuracy

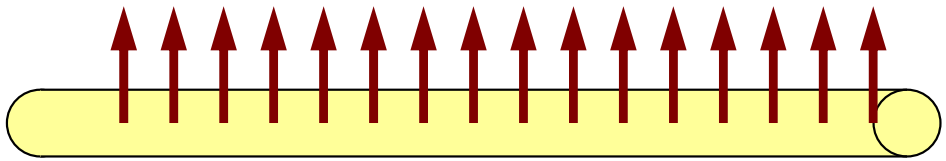
- Finer Meshes
- Higher Order Shape/Interpolation Function（高次補間関数・形状関数）
  - Higher-Order Element（高次要素）
  - Linear-Element, 1st-Order Element: Lower Order（低次要素）
- Formulation which assures continuity of n-th order derivatives
  - C^n Continuity（C^n連続性）

Remedies for Higher Accuracy

- Finer Meshes
- Higher Order Shape/Interpolation Function（高次補間関数・形状関数）
  - Higher-Order Element（高次要素）
  - Linear-Element, 1st-Order Element: Lower Order（低次要素）
- Formulation which assures continuity of n-th order derivatives
  - C^n Continuity（C^n連続性）
- Linear Elements
  - Piecewise Linear
  - C^0 Continuity
    - Only dependent variables are continuous at element boundary

Example: 1D Heat Transfer (1/2)





Convective Heat Transfer on Cylindrical Surface

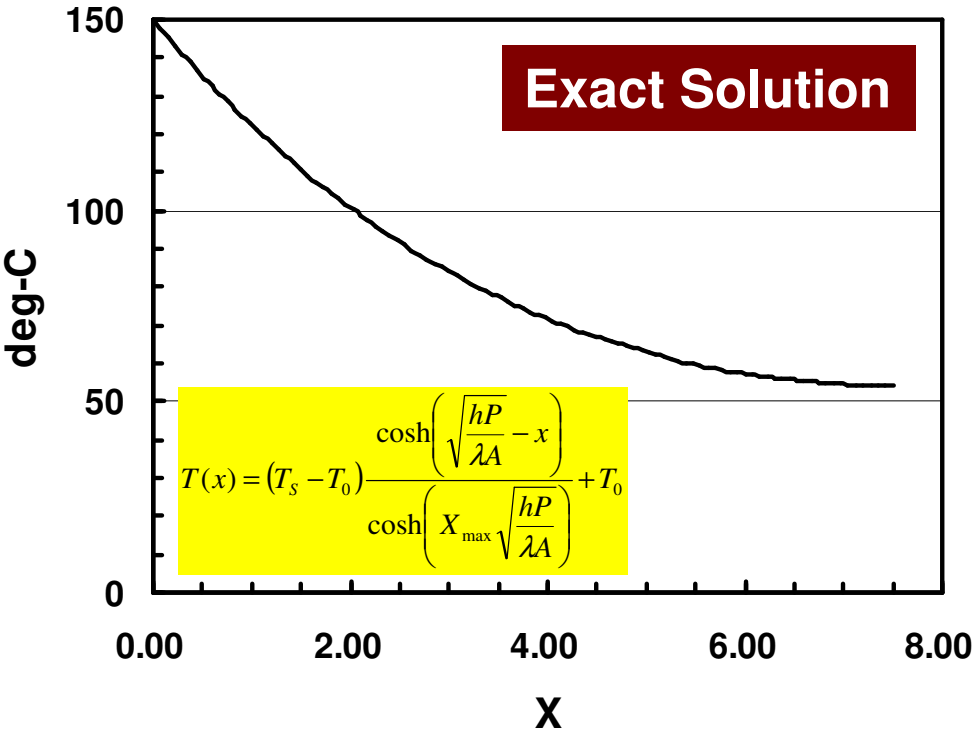
- Temp. Thermal Fins
- Circular Sectional Area, r=1cm
- Boundary Condition
  - x=0 : Fixed Temperature
  - x=7.5 : Insulated
- Convective Heat Transfer on Cylindrical Surface
  - $q = h\,(T-T_0)$
  - q : Heat Flux
    - Heat Flow/Unit Surface Area/sec.

Example: 1D Heat Transfer (2/2)

```
### RESULTS (linear interpolation)
ID        X          FEM.       ANALYTICAL
 1      0.00000   150.00000    150.00000     ERR(%):  0.00000
 2      1.87500   102.62226    103.00165     ERR(%):  0.25292
 3      3.75000    73.82803     74.37583     ERR(%):  0.36520
 4      5.62500    58.40306     59.01653     ERR(%):  0.40898
 5      7.50000    53.55410     54.18409     ERR(%):  0.41999
```

```
### RESULTS (quadratic interpolation)
ID        X          FEM.       ANALYTICAL
 1      0.00000   150.00000    150.00000     ERR(%):  0.00000
 2      1.87500   102.98743    103.00165     ERR(%):  0.00948
 3      3.75000    74.40203     74.37583     ERR(%):  0.01747
 4      5.62500    59.02737     59.01653     ERR(%):  0.00722
 5      7.50000    54.21426     54.18409     ERR(%):  0.02011
```

```
### RESULTS (linear interpolation)
ID        X          FEM.       ANALYTICAL
 1      0.00000   150.00000    150.00000     ERR(%):  0.00000
 2      0.93750   123.71561    123.77127     ERR(%):  0.03711
 3      1.87500   102.90805    103.00165     ERR(%):  0.06240
 4      2.81250    86.65618     86.77507     ERR(%):  0.07926
 5      3.75000    74.24055     74.37583     ERR(%):  0.09019
 6      4.68750    65.11151     65.25705     ERR(%):  0.09703
 7      5.62500    58.86492     59.01653     ERR(%):  0.10107
 8      6.56250    55.22426     55.37903     ERR(%):  0.10317
 9      7.50000    54.02836     54.18409     ERR(%):  0.10382
```

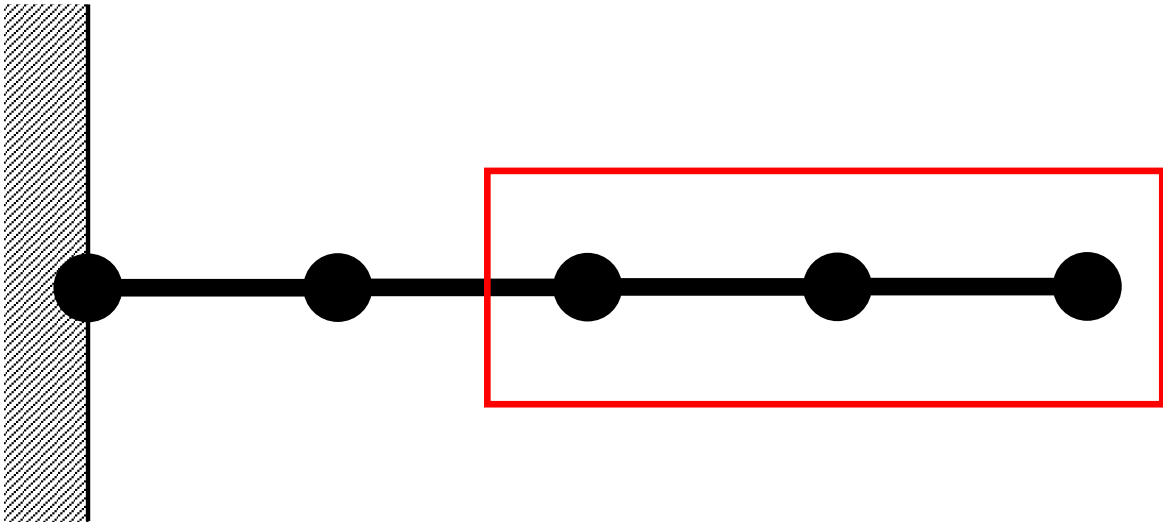


Quadratic interpolation provides more accurate solution, especially if X is close to 7.50cm.

1D Quadratic Element (1/2)

一次元二次要素

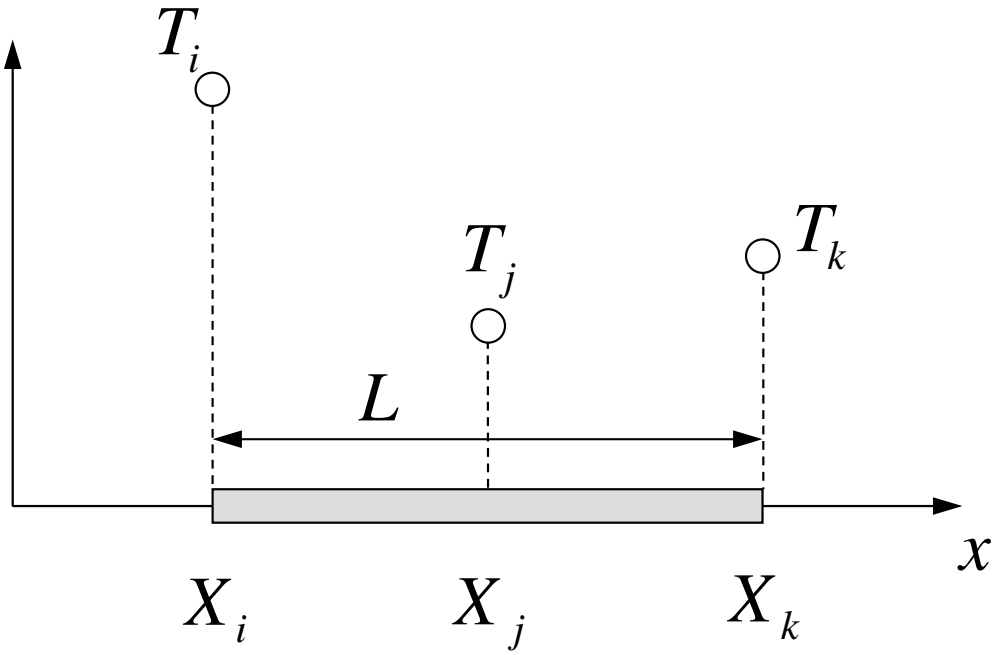
- Length= L
- (i,k): Both Ends
- (j): Intermediate Node
  - ✓ Mid-Point（中間節点）



- Distribution of T in each element:

$$T = \alpha_1 + \alpha_2 x + \alpha_3 x^2$$

$$T_i = \alpha_1 + X_i \alpha_2 + X_i^2 \alpha_3$$
$$T_j = \alpha_1 + X_j \alpha_2 + X_j^2 \alpha_3$$
$$T_k = \alpha_1 + X_k \alpha_2 + X_k^2 \alpha_3$$



1D Quadratic Element (1/2)

一次元二次要素

- Length= L
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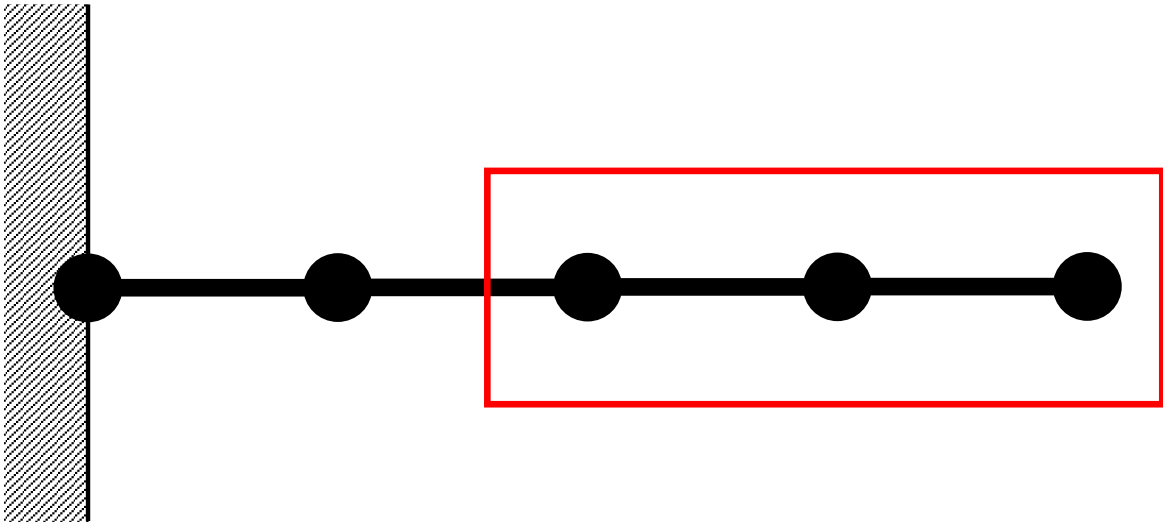
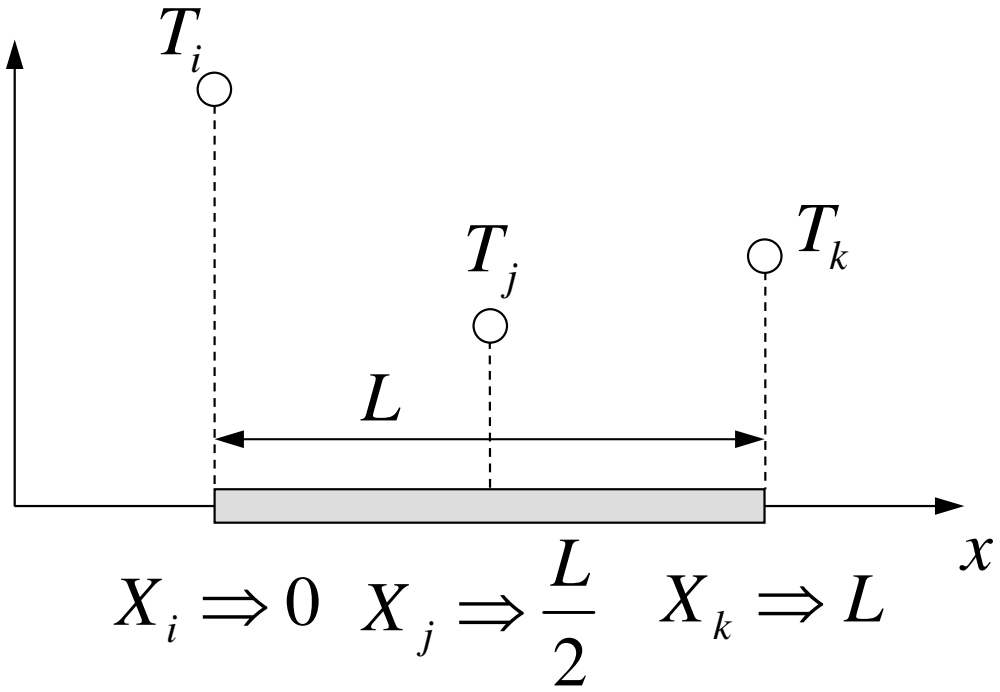


- Distribution of T in each element:

$$T = \alpha_1 + \alpha_2 x + \alpha_3 x^2$$

$$u_i = \alpha_1, \quad u_j = \alpha_1 + \frac{L}{2}\alpha_2 + \frac{L^2}{4}\alpha_3$$

$$u_k = \alpha_1 + L\alpha_2 + L^2\alpha_3$$

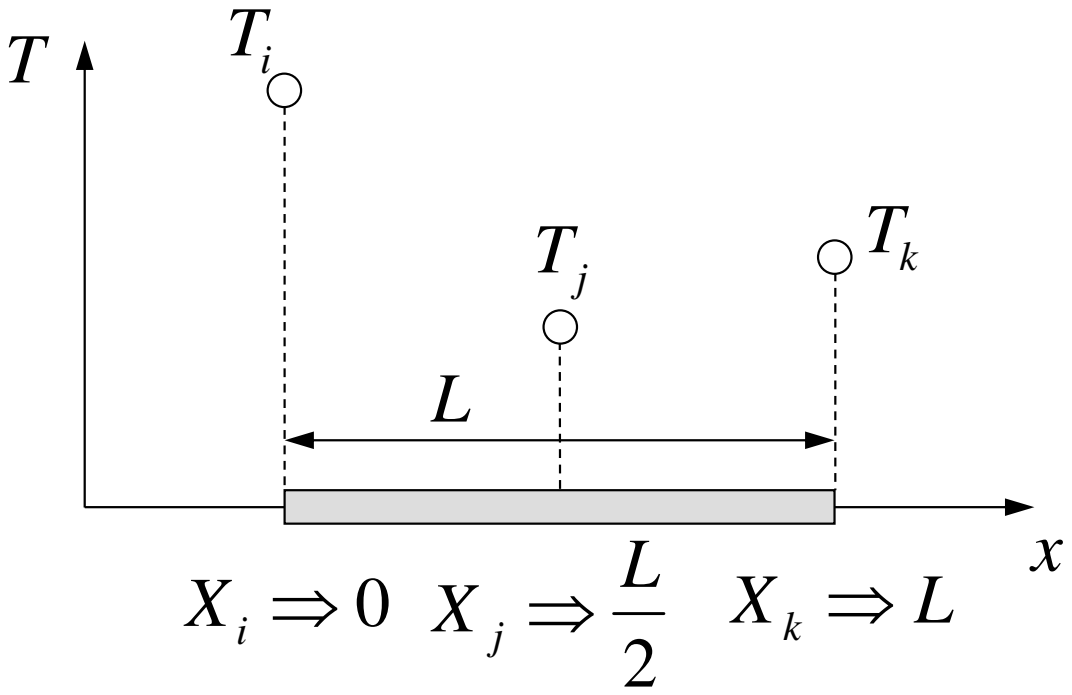


1D Quadratic Element (2/2)

一次元二次要素

- Coef's are calculated based on info. at each node:

$$\alpha_1 = T_i,\ \alpha_2 = \frac{4T_i - 3T_j - T_k}{L},$$
$$\alpha_3 = \frac{2}{L^2}\left(T_i - 2T_j + T_k\right)$$

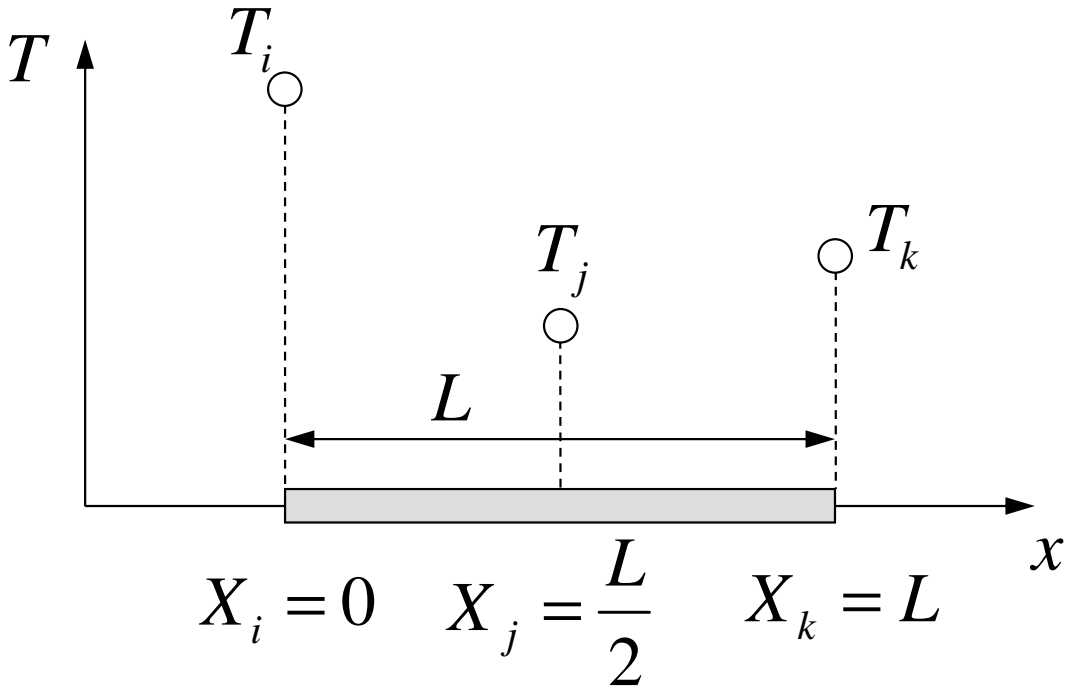


- Shape Functions: N_i, N_j, N_k

$$T = N_i T_i + N_j T_j + N_k T_k$$
$$= \left(1 - \frac{2x}{L}\right)\left(1 - \frac{x}{L}\right)T_i + \left(\frac{4x}{L}\right)\left(1 - \frac{x}{L}\right)T_j + \left(-\frac{x}{L}\right)\left(1 - \frac{2x}{L}\right)T_k$$

1D Quadratic Element

一次元二次要素



Intermediate Node
Mid Point: j

Integration over Each Element: $[k]$ (1/2)

$$N_i = \left(1-\frac{2x}{L}\right)\left(1-\frac{x}{L}\right) \qquad \frac{dN_i}{dx} = \left(\frac{4x}{L^2}-\frac{3}{L}\right)$$

$$N_j = \left(\frac{4x}{L}\right)\left(1-\frac{x}{L}\right) \qquad \frac{dN_j}{dx} = \left(\frac{4}{L}-\frac{8x}{L^2}\right)$$

$$N_k = \left(-\frac{x}{L}\right)\left(1-\frac{2x}{L}\right) \qquad \frac{dN_k}{dx} = \left(\frac{4x}{L^2}-\frac{1}{L}\right)$$

Integration over Each Element: $[k]$ (2/2)

$$\int_V \lambda \left(\frac{d[N]^T}{dx} \frac{d[N]}{dx} \right) dV = \int_0^L \begin{bmatrix} dN_i/dx \\ dN_j/dx \\ dN_k/dx \end{bmatrix} \lambda \left[\frac{dN_i}{dx}, \frac{dN_j}{dx}, \frac{dN_k}{dx} \right] A\, dx$$

$$= \lambda A \int_0^L \begin{bmatrix} \frac{dN_i}{dx}\frac{dN_i}{dx} & \frac{dN_i}{dx}\frac{dN_j}{dx} & \frac{dN_i}{dx}\frac{dN_k}{dx} \\ \frac{dN_j}{dx}\frac{dN_i}{dx} & \frac{dN_j}{dx}\frac{dN_j}{dx} & \frac{dN_j}{dx}\frac{dN_k}{dx} \\ \frac{dN_k}{dx}\frac{dN_i}{dx} & \frac{dN_k}{dx}\frac{dN_j}{dx} & \frac{dN_k}{dx}\frac{dN_k}{dx} \end{bmatrix} dx = \frac{\lambda A}{6L} \begin{bmatrix} +14 & -16 & +2 \\ -16 & +32 & -16 \\ +2 & -16 & +14 \end{bmatrix}$$

Integration over Each Element: $\{f\}$

$$\int_V \dot{Q}[N]^T dV = \dot{Q}A\int_0^L \begin{bmatrix} N_i \\ N_j \\ N_k \end{bmatrix} dx = \dot{Q}A\int_0^L \begin{bmatrix} 1-\frac{3x}{L}+\frac{2x^2}{L^2} \\ \frac{4x}{L}-\frac{4x^2}{L^2} \\ -\frac{x}{L}+\frac{2x^2}{L^2} \end{bmatrix} dx = \frac{\dot{Q}AL}{6}\begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix}$$

1 : 4 : 1

The Ratio was 1:1 in Linear Element

$$N_i = \left(\frac{X_j - x}{L}\right), \quad N_j = \left(\frac{x - X_i}{L}\right) \qquad \frac{dN_i}{dx} = \left(\frac{-1}{L}\right), \quad \frac{dN_j}{dx} = \left(\frac{1}{L}\right)$$

$$\int_V \dot{Q}[N]^T dV = \dot{Q}A\int_0^L \begin{bmatrix} 1 - x/L \\ x/L \end{bmatrix} dx = \frac{\dot{Q}AL}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Integration over Each Element: $\{f\}$

$$\int_V \dot{Q}[N]^T dV = \dot{Q}A\int_0^L \begin{bmatrix} N_i \\ N_j \\ N_k \end{bmatrix} dx = \dot{Q}A\int_0^L \begin{bmatrix} 1-\frac{3x}{L}+\frac{2x^2}{L^2} \\ \frac{4x}{L}-\frac{4x^2}{L^2} \\ -\frac{x}{L}+\frac{2x^2}{L^2} \end{bmatrix} dx = \frac{\dot{Q}AL}{6}\begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix}$$

1 : 4 : 1

Volume Heat Flux

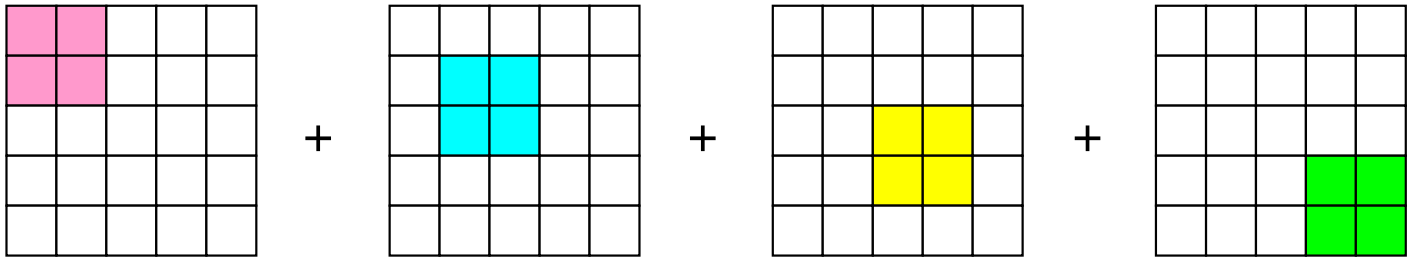
$$\int_S \bar{q}[N]^T dS = \bar{q}A\big|_{x=L} = \bar{q}A\begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$$

Surface Heat Flux

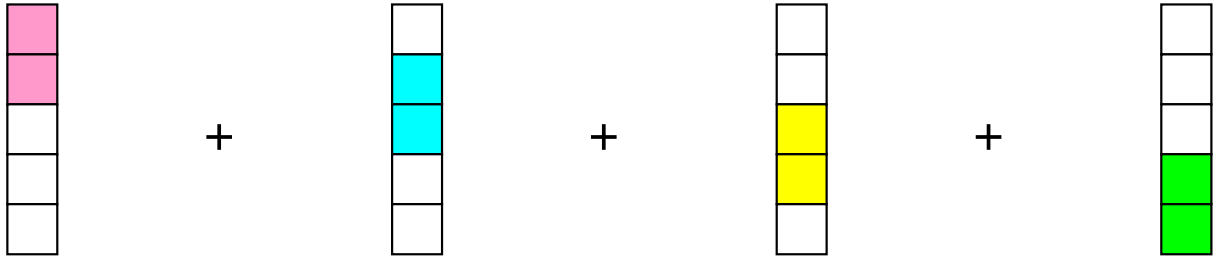
Element Eqn's/Accumulation

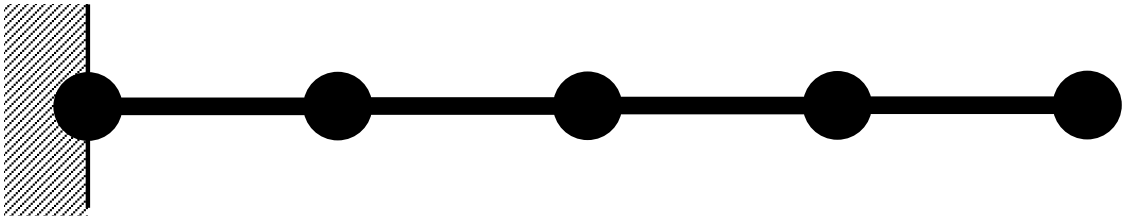
1D Linear Element

$$[K]=\sum_{i=1}^{4}\left[k^{(i)}\right]=$$



$$\{F\}=\sum_{i=1}^{4}\left\{f^{(i)}\right\}=$$

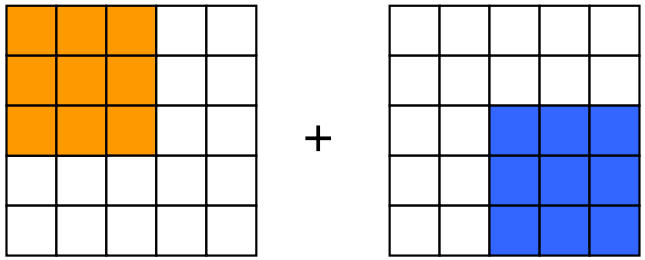




Element Eqn's/Accumulation

1D Quadratic Element, 2 Elements

$$[K] = \sum_{i=1}^{2} [k^{(i)}] =$$



$$\{F\} = \sum_{i=1}^{4} \{f^{(i)}\} =$$