Introduction to Parallel FEM in C
Parallel Data Structure

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Technical & Scientific Computing II (4820-1028)

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Hybrid Distributed Parallel Computing (3747-111)

Parallel Computing

- Faster, Larger & More Complicated

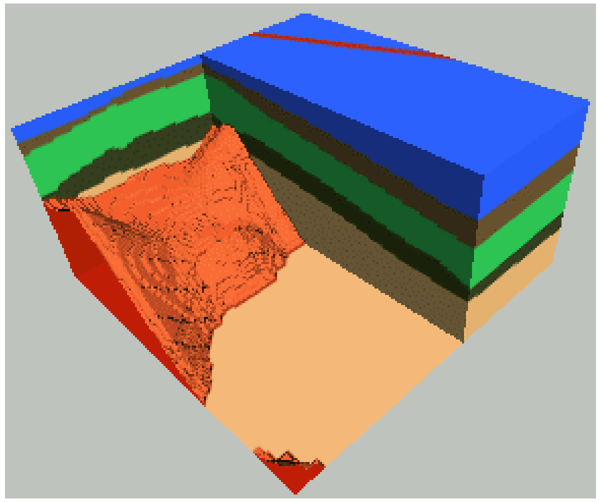
- Scalability
  - Solving N^x scale problem using N^x computational resources during same computation time
    - for large-scale problems: **Weak Scaling**
    - e.g. CG solver: more iterations needed for larger problems
  - Solving a problem using N^x computational resources during 1/N computation time
    - for faster computation: **Strong Scaling**

What is Parallel Computing ? (1/2)

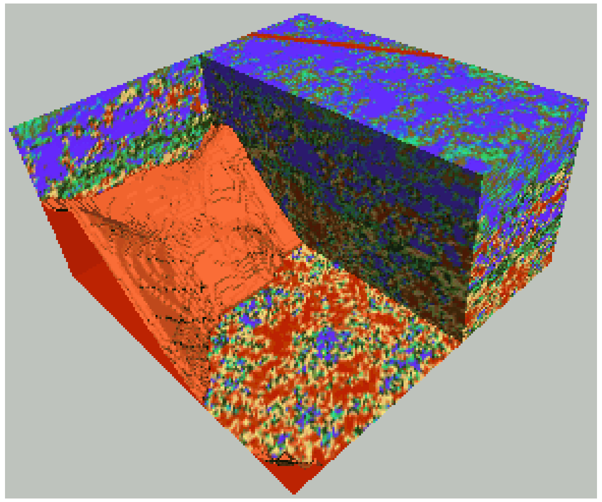
- to solve larger problems faster

Homogeneous/Heterogeneous Porous Media

Lawrence Livermore National Laboratory



Homogeneous

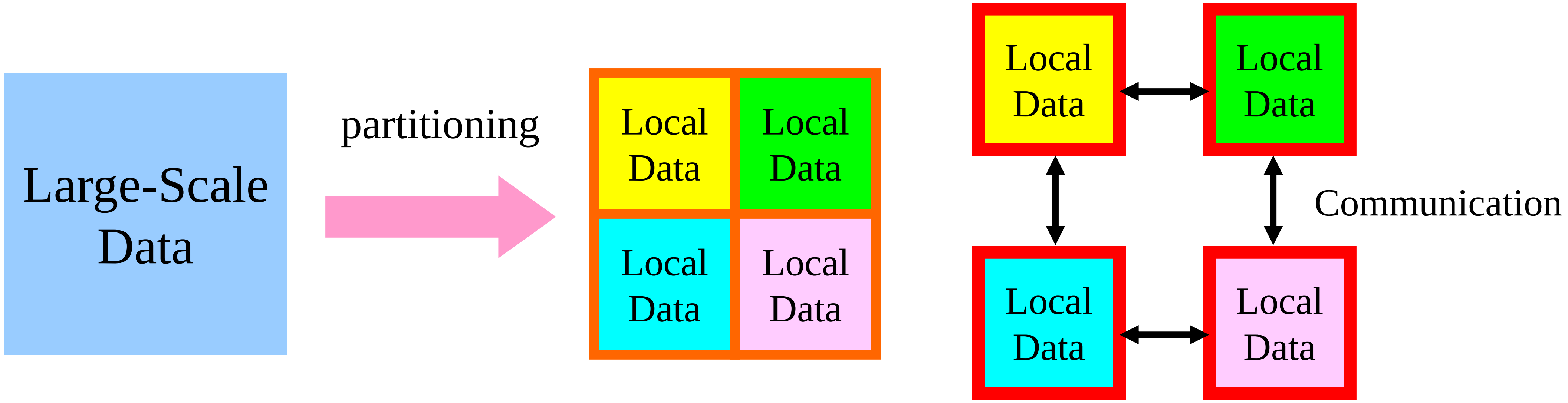


Heterogeneous

very fine meshes are required for simulations of heterogeneous field.

What is Parallel Computing ? (2/2)

- PC with 1GB memory : 1M meshes are the limit for FEM
  - Southwest Japan with 1,000km x 1,000km x 100km in 1km mesh -> 10^8 meshes
- Large Data -> Domain Decomposition -> Local Operation
- Inter-Domain Communication for Global Operation



What is Communication ?

- Parallel Computing -> Local Operations

- Communications are required in Global Operations for Consistency.

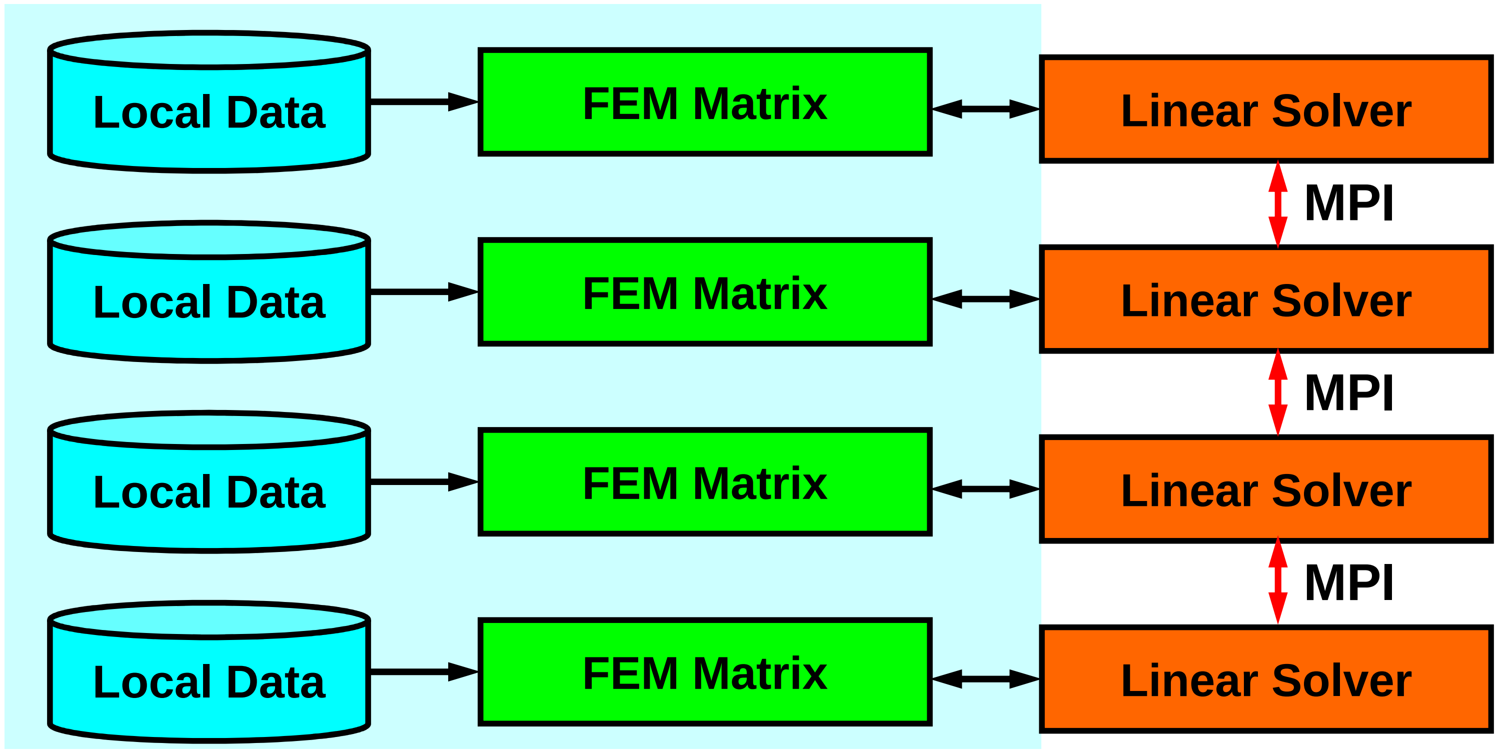
Operations in Parallel FEM

SPMD: Single-Program Multiple-Data

Large Scale Data -> partitioned into Distributed Local Data Sets.

FEM code can assembles coefficient matrix for each local data set : this part could be completely local, same as serial operations

Global Operations & Communications happen only in Linear Solvers dot products, matrix-vector multiply, preconditioning



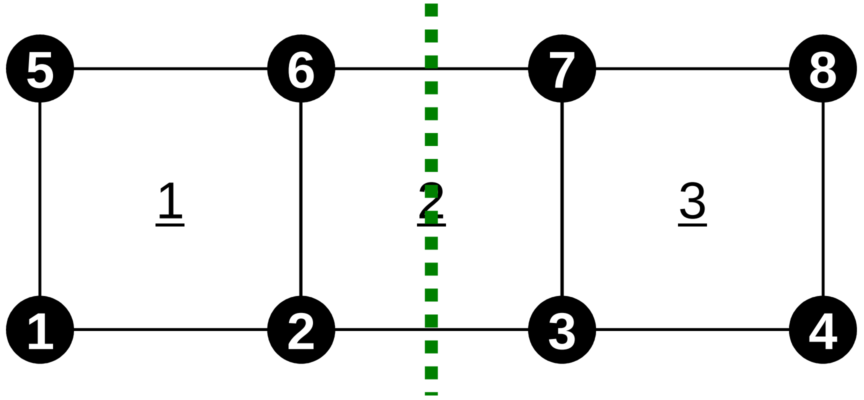
Parallel FEM Procedures

- Design on “Local Data Structure” is important
  - for SPMD-type operations in the previous page

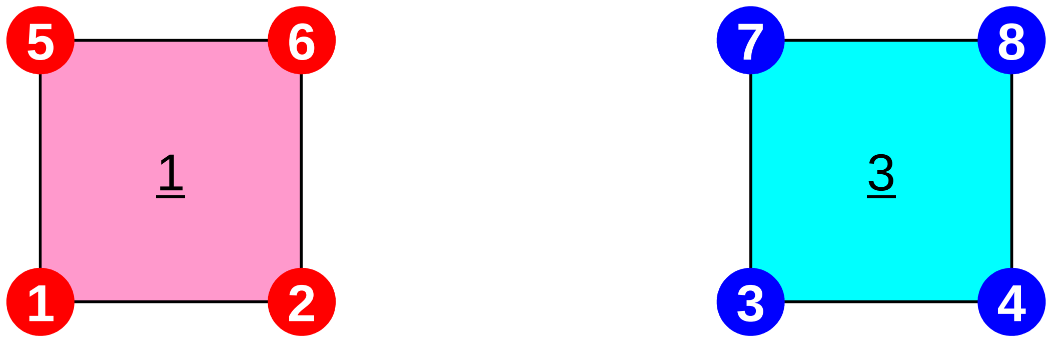
- Matrix Generation
- Preconditioned Iterative Solvers for Linear Equations

Bi-Linear Square Elements

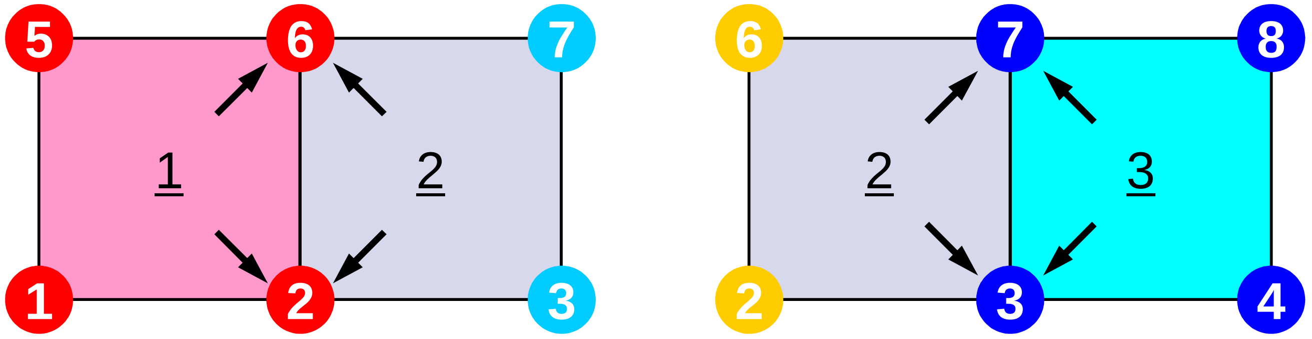
Values are defined on each node



divide into two domains by “node-based” manner, where number of “nodes (vertices)” are balanced.



Local information is not enough for matrix assembling.



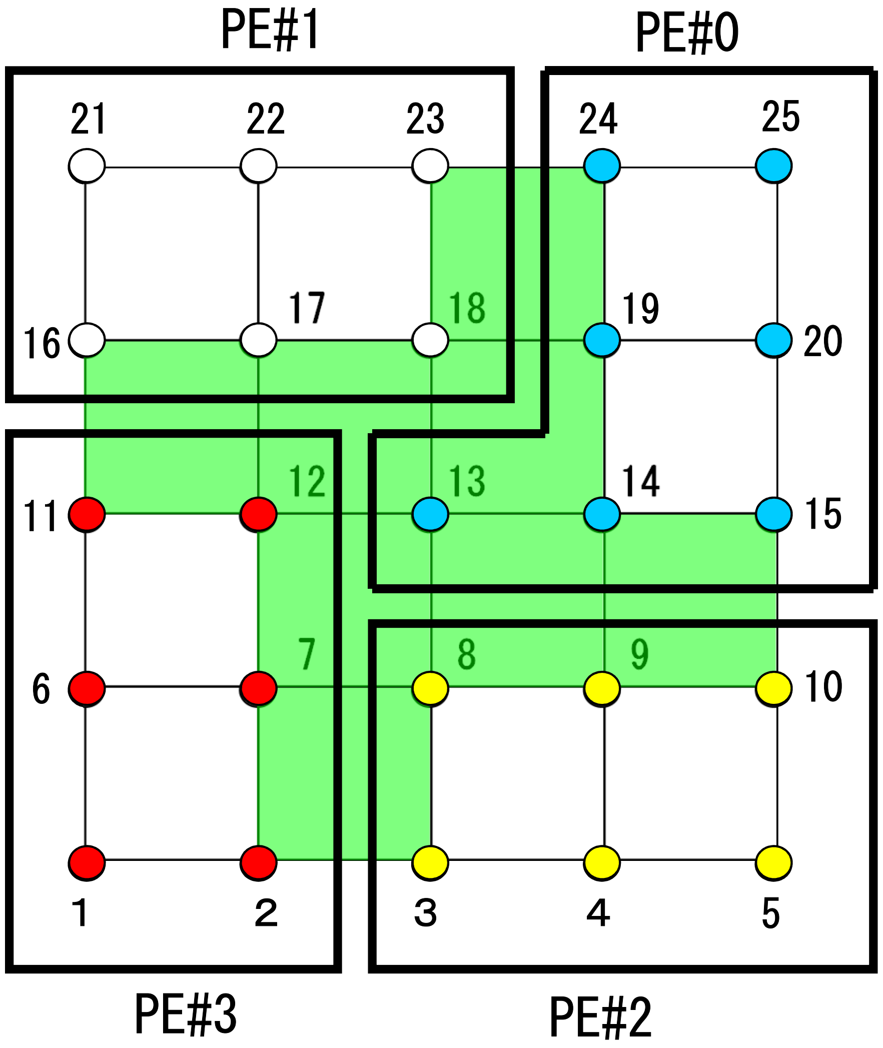
Information of overlapped elements and connected nodes are required for matrix assembling on boundary nodes.

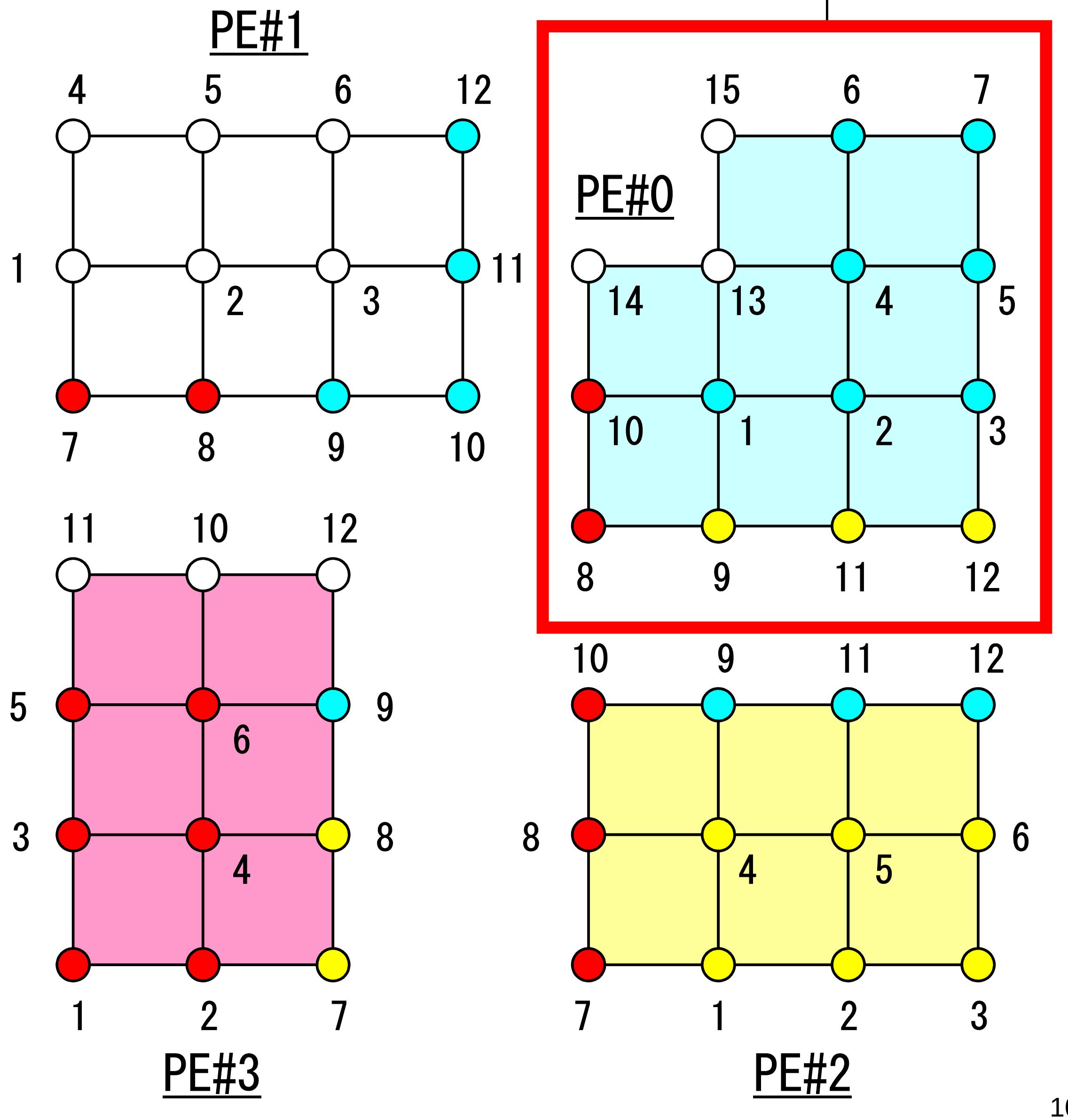
Local Data of Parallel FEM

- Node-based partitioning for IC/ILU type preconditioning methods
- Local data includes information for :
  - Nodes originally assigned to the partition/PE
  - Elements which include the nodes : Element-based operations (Matrix Assemble) are allowed for fluid/structure subsystems.
  - All nodes which form the elements but out of the partition
- Nodes are classified into the following 3 categories from the viewpoint of the message passing
  - Internal nodes originally assigned nodes
  - External nodes in the overlapped elements but out of the partition
  - Boundary nodes *external nodes* of other partition
- Communication table between partitions
- NO global information required except partition-to-partition connectivity

Node-based Partitioning

internal nodes - elements - external nodes



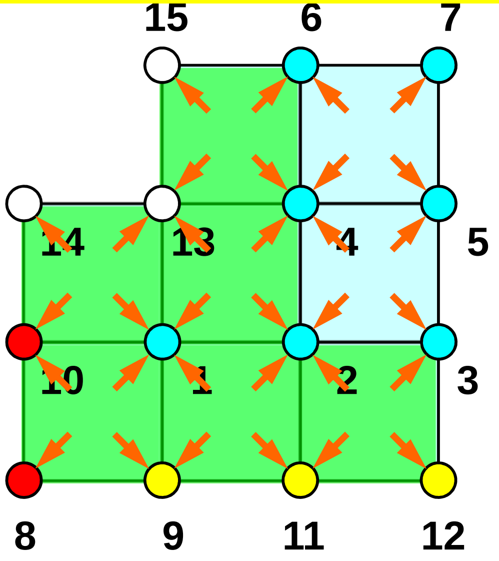


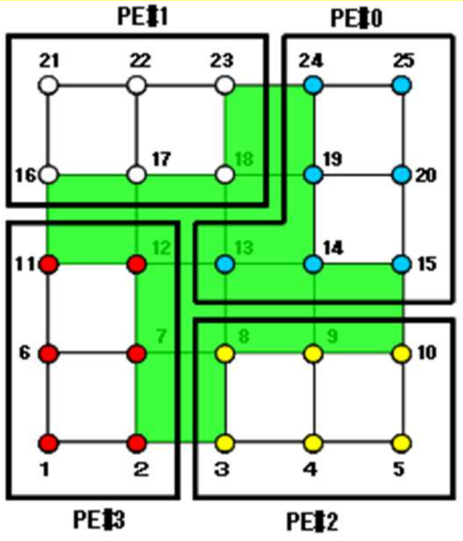
Node-based Partitioning

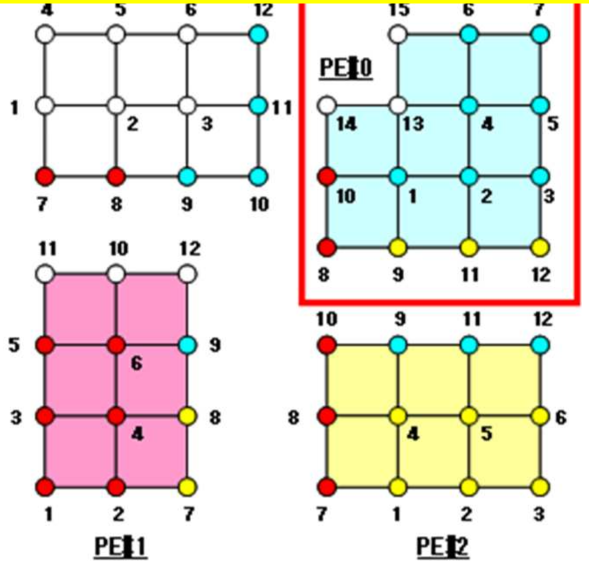
internal nodes - elements - external nodes

- Partitioned nodes themselves (<u>Internal Nodes</u>)　内点
- Elements which include Internal Nodes　内点を含む要素
- <u>External Nodes</u> included in the Elements　外点

  in overlapped region among partitions.
- Info of External Nodes are required for completely local element–based operations on each processor.







We do not need communication during matrix assemble !!

- Partitioned nodes themselves (Internal Nodes)
- Elements which include Internal Nodes
- External Nodes included in the Elements
  in overlapped region among partitions.
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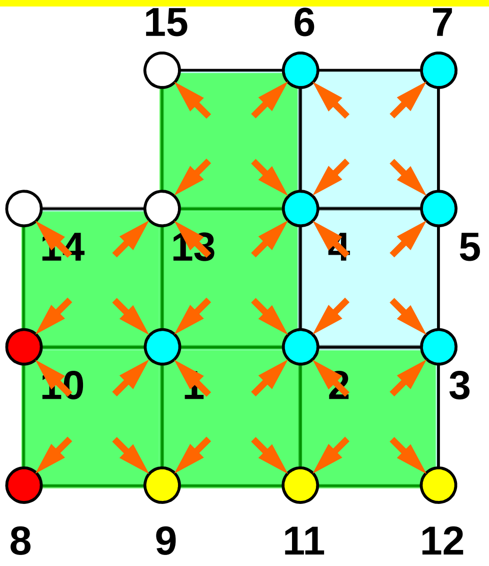
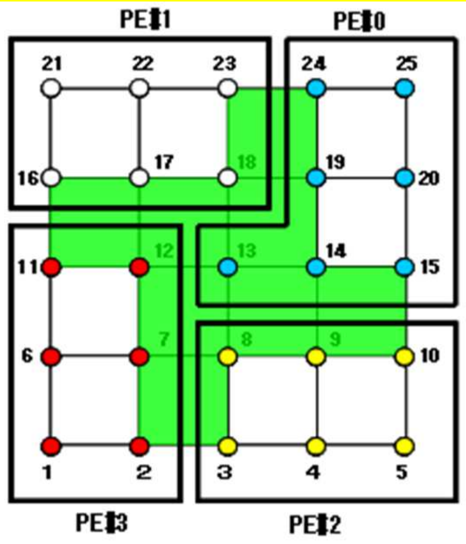
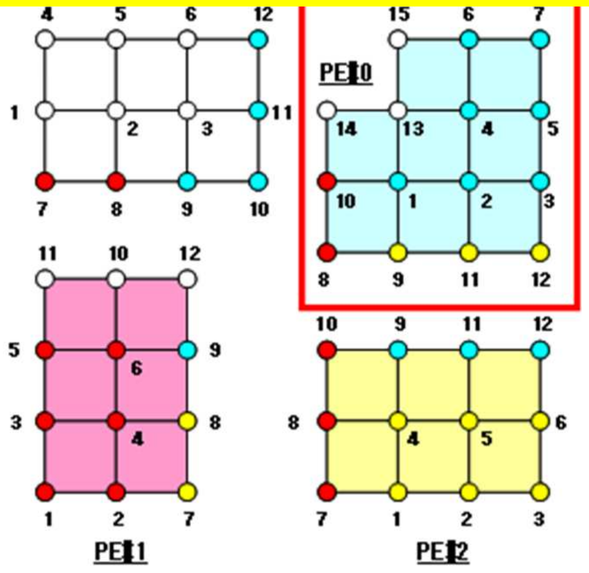
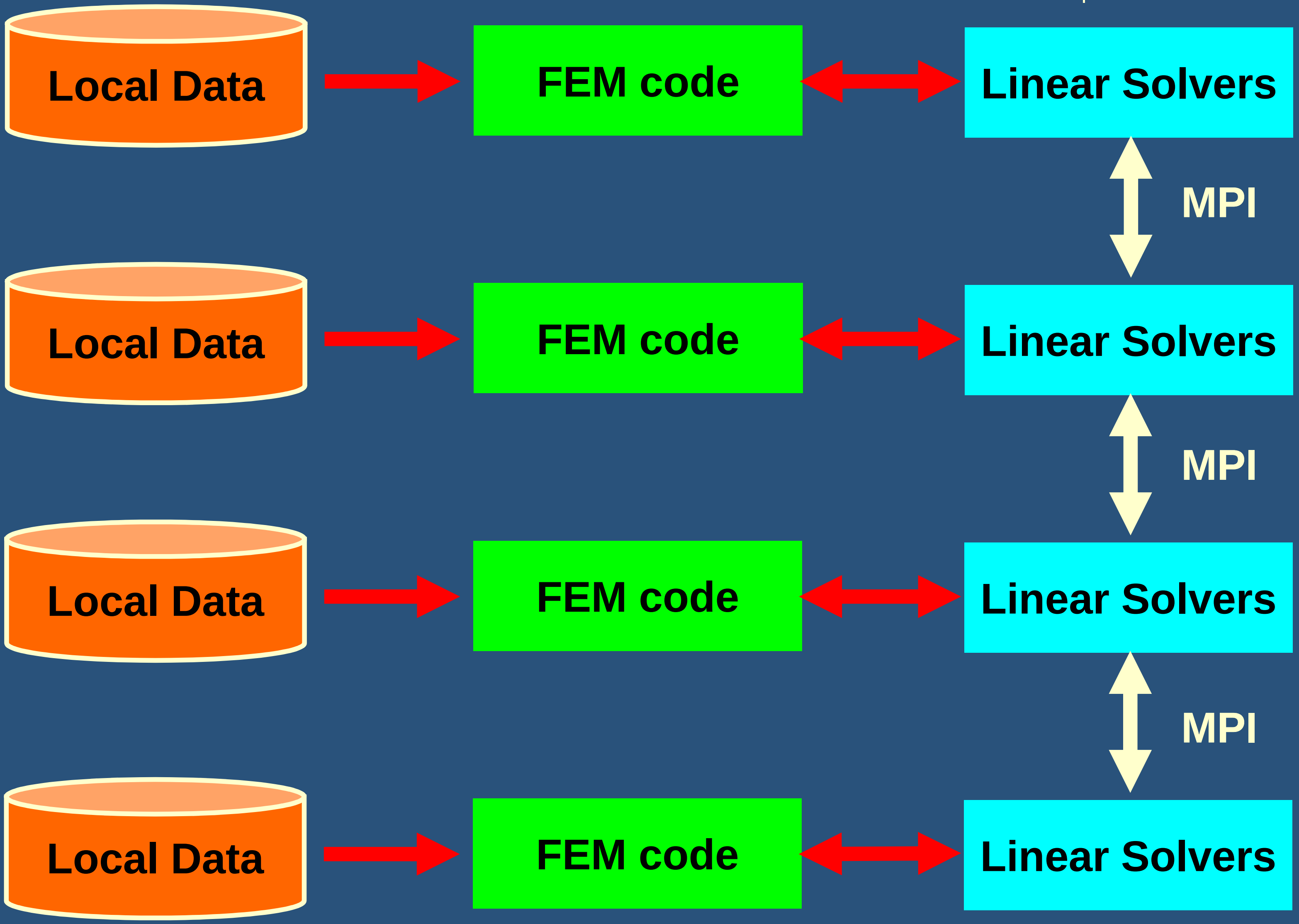






Parallel Computing in FEM

SPMD: Single-Program Multiple-Data



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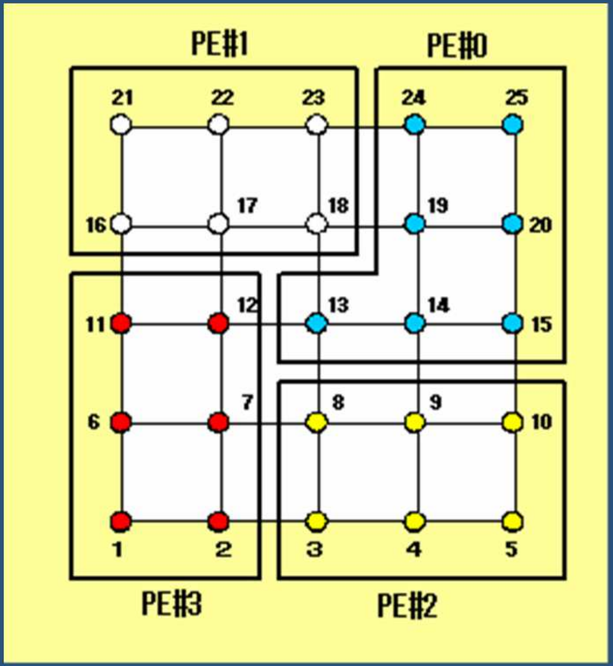

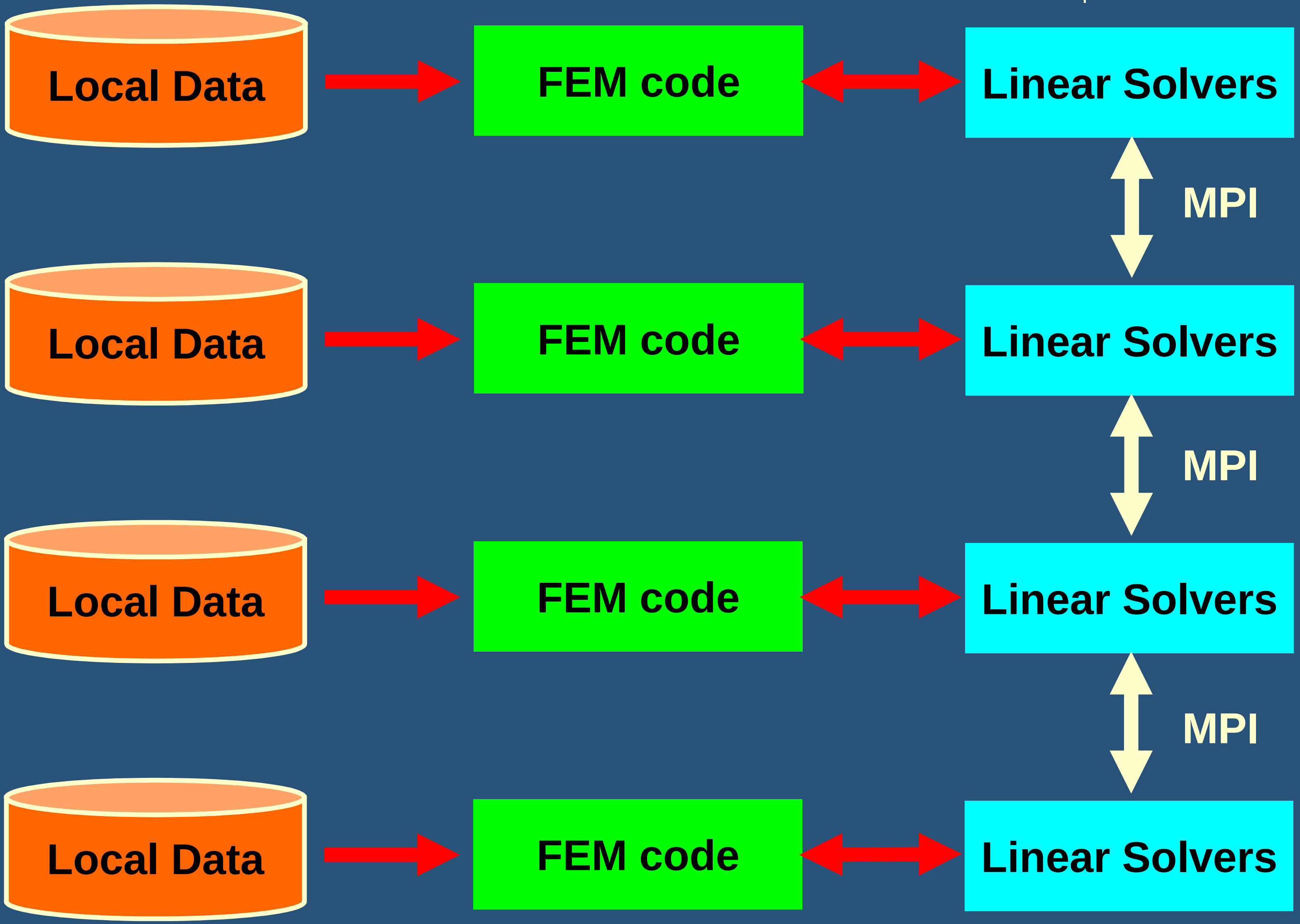

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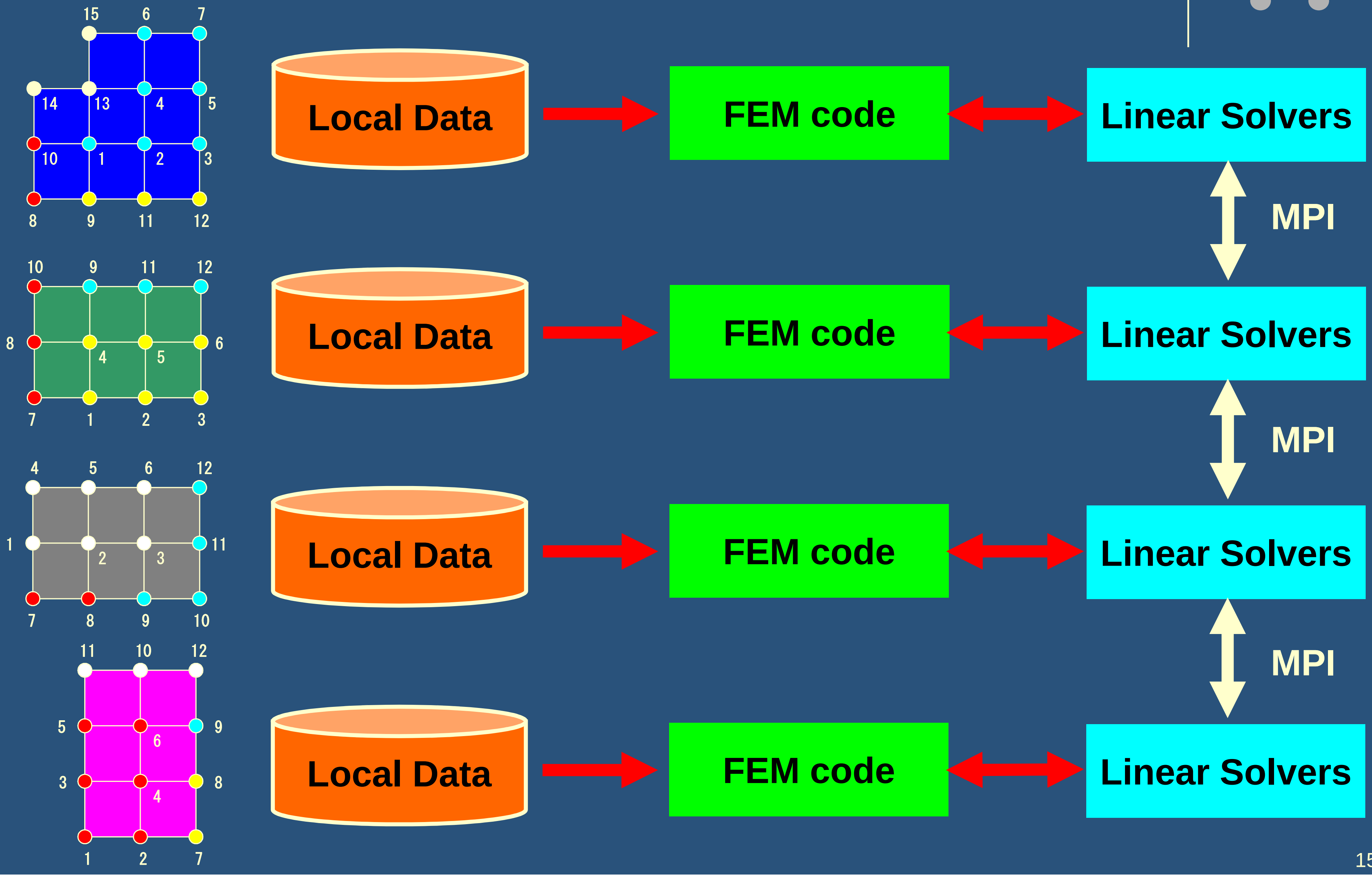

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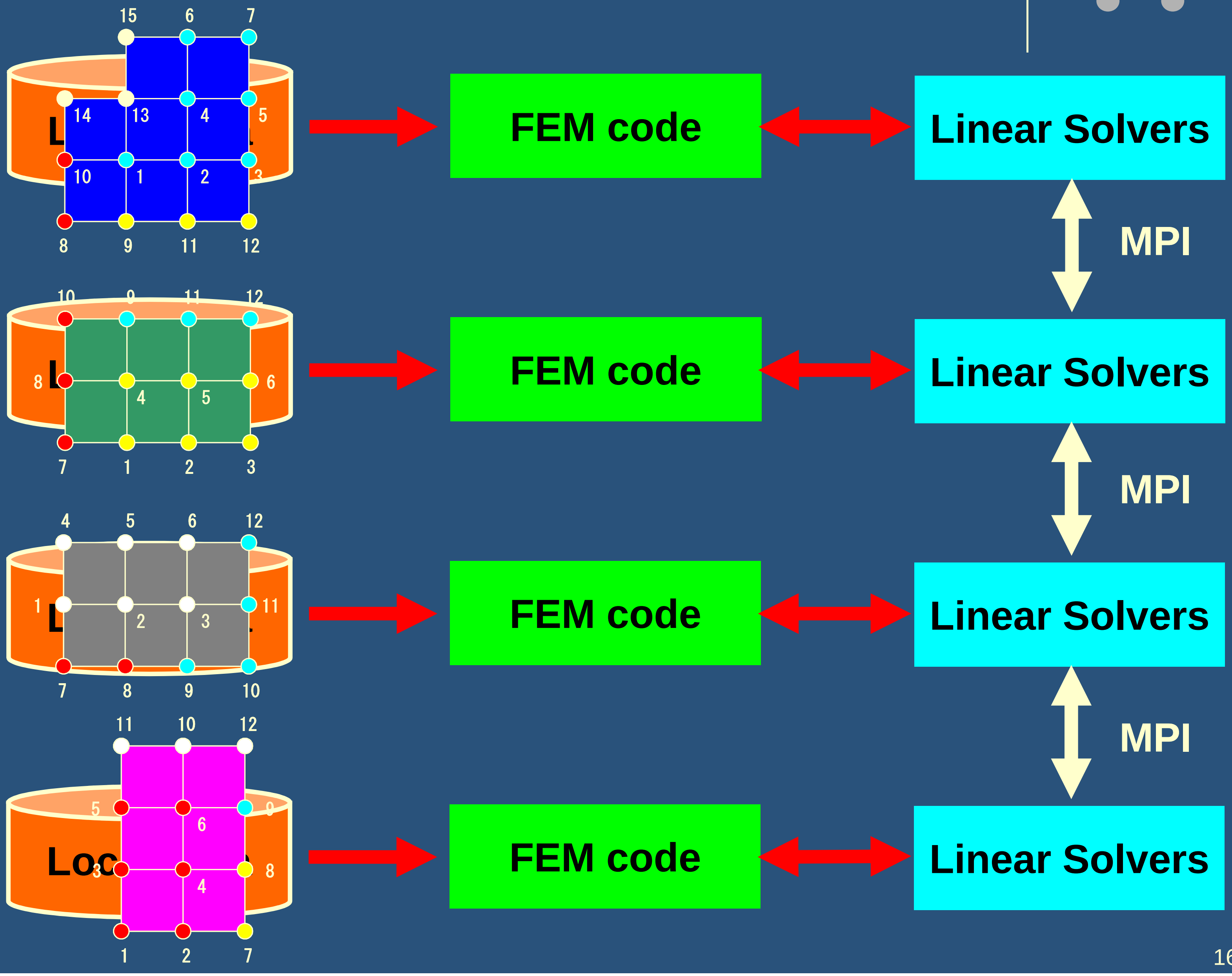

Parallel Computing in FEM

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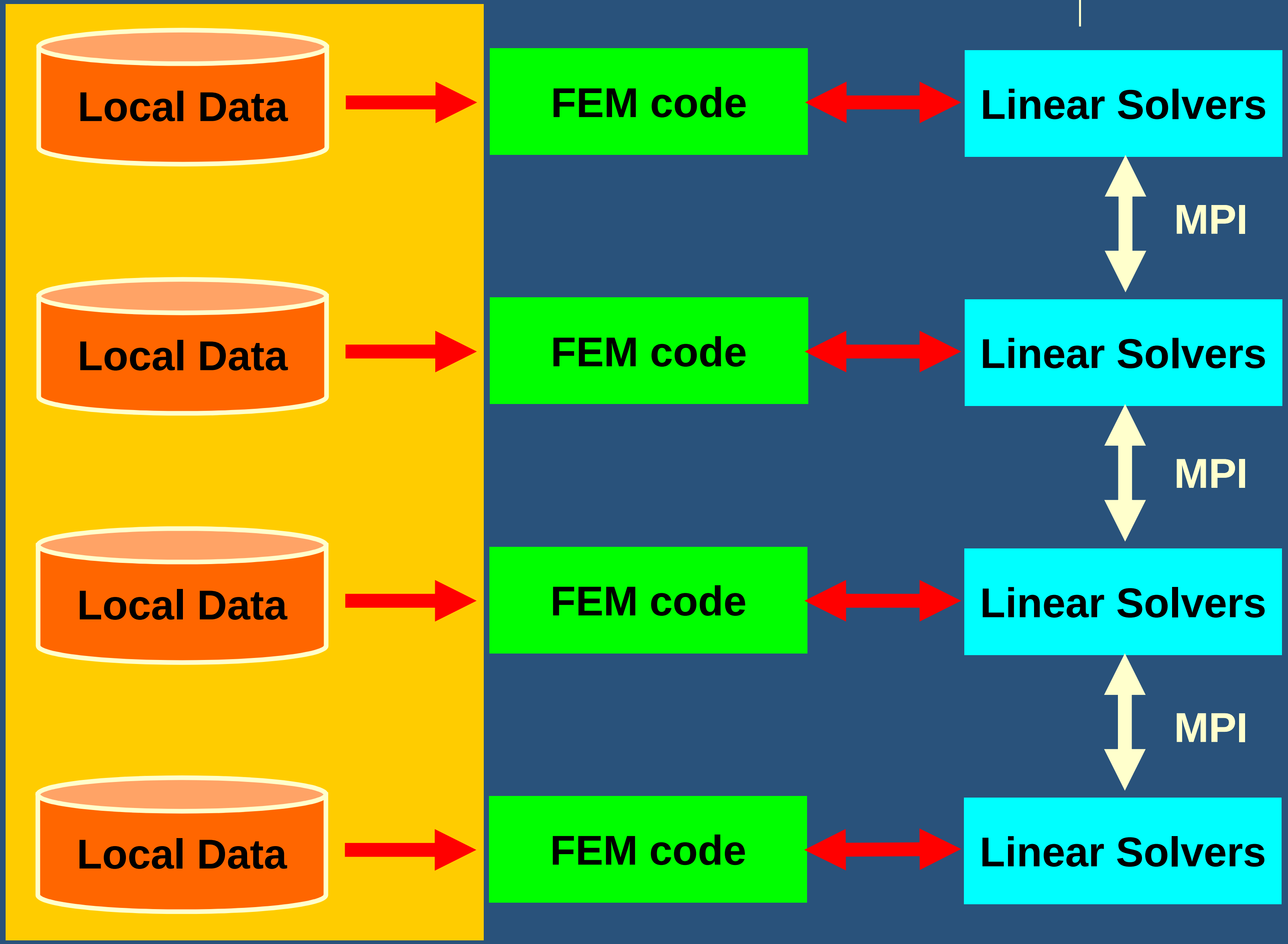
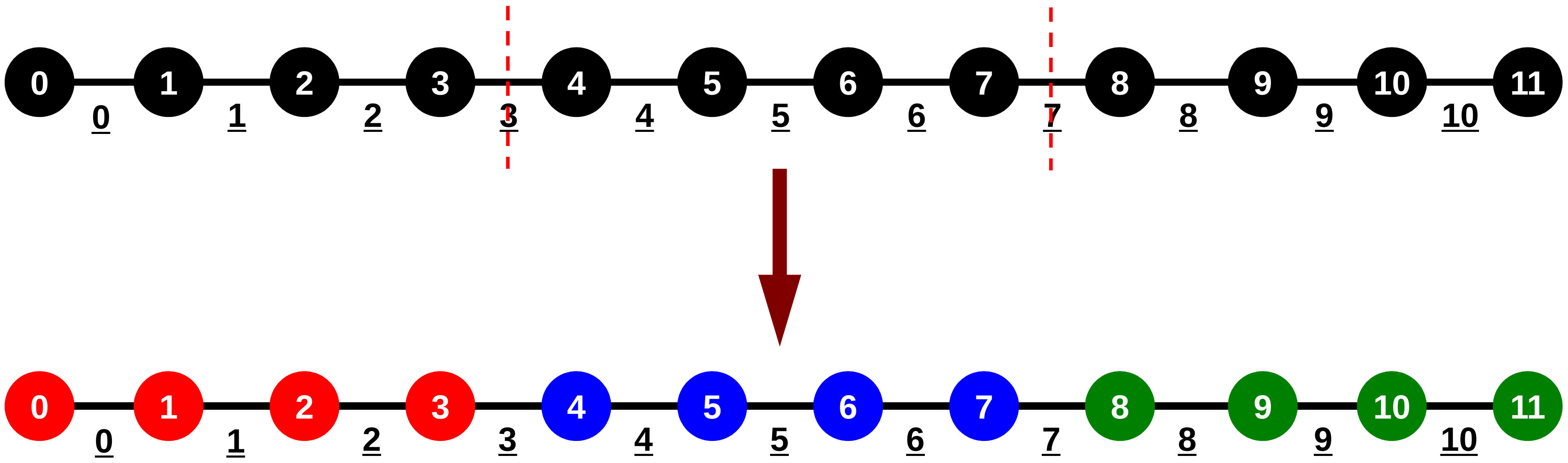


What is Communications ?

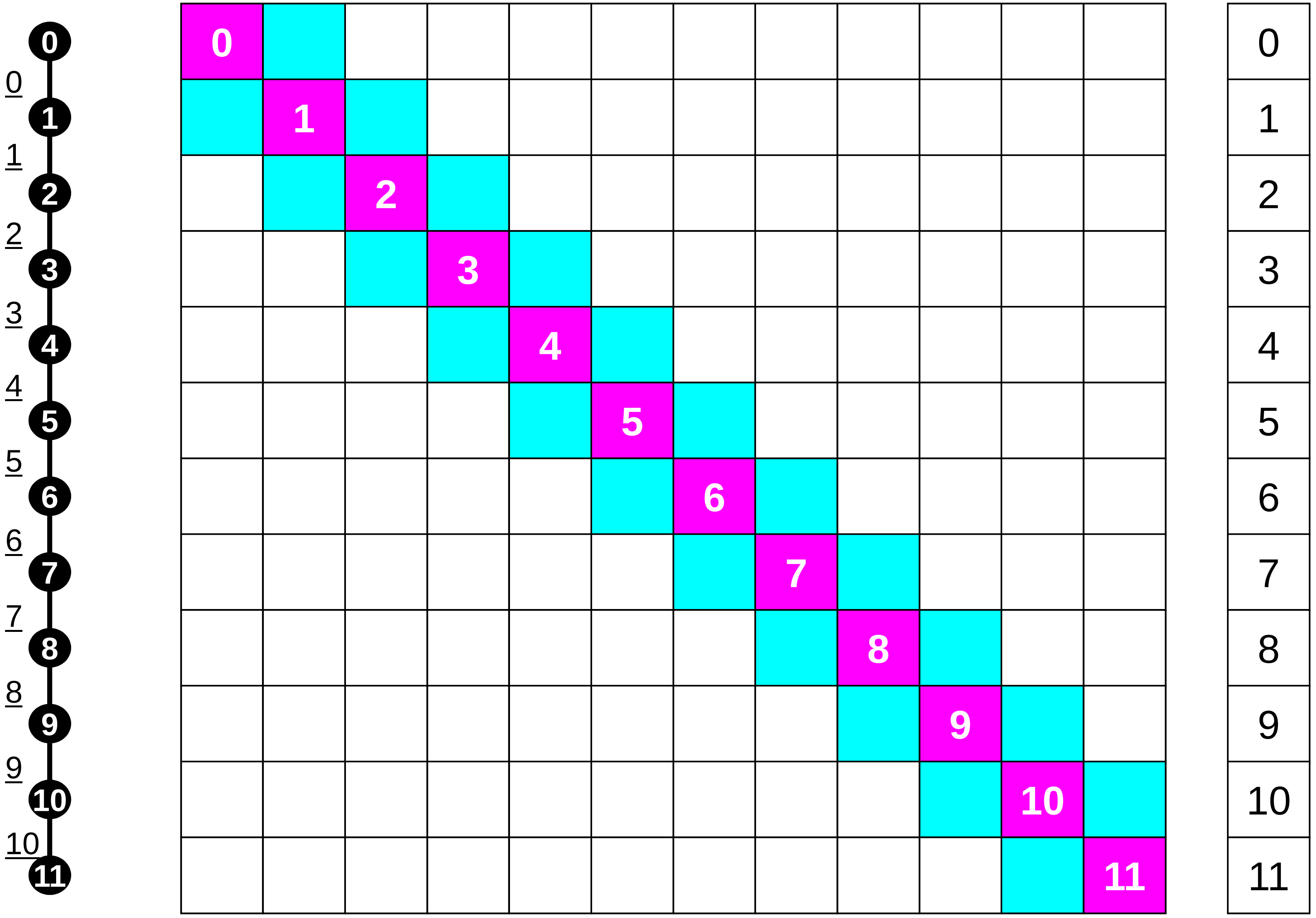
- to get information of “external nodes” from external partitions (local data)
- “Communication tables” contain the information

1D FEM: 12 nodes/11 elem's/3 domains

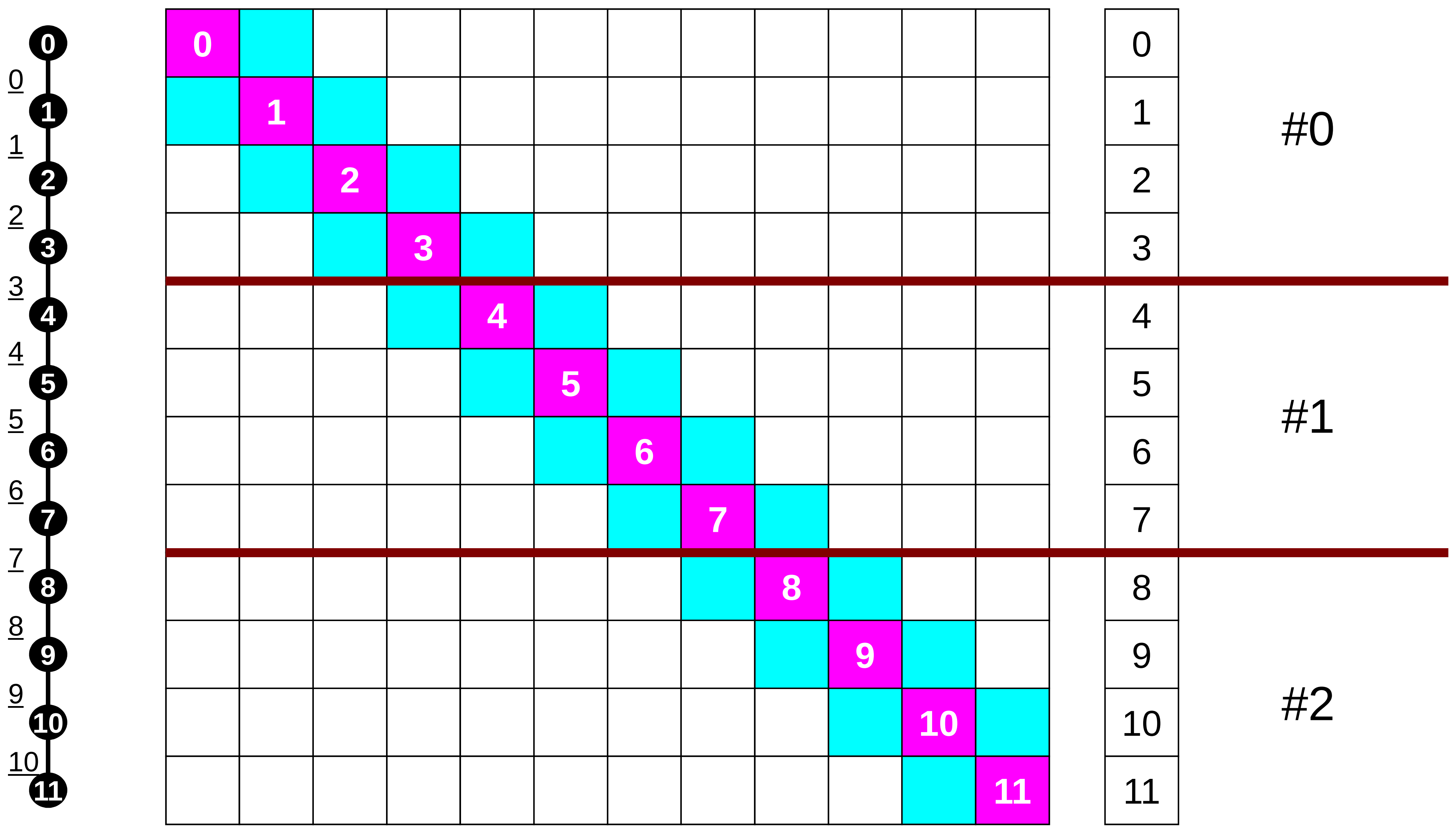


1D FEM: 12 nodes/11 elem’s/3 domains

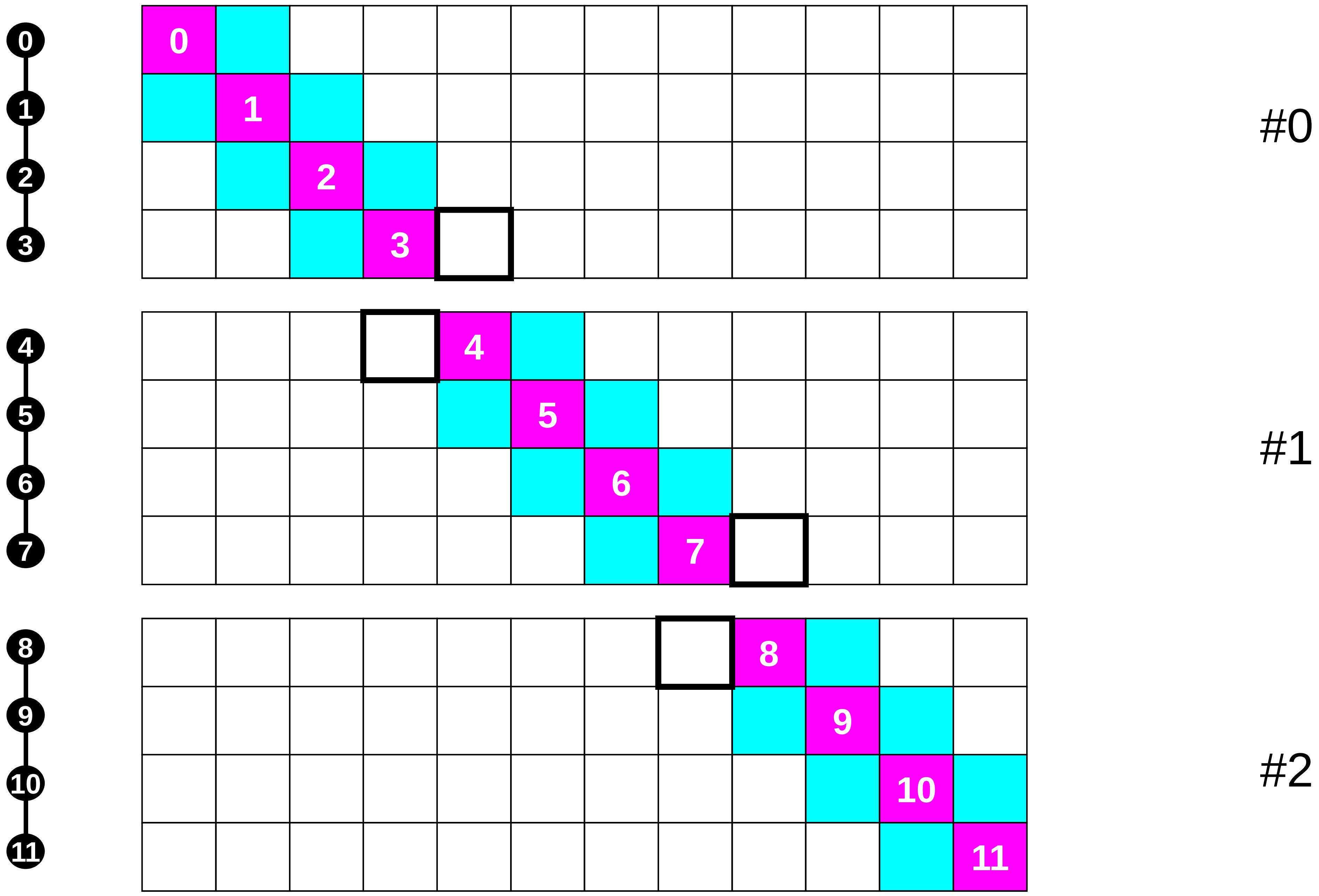
三重対角行列：Tri-Diagonal Matrix



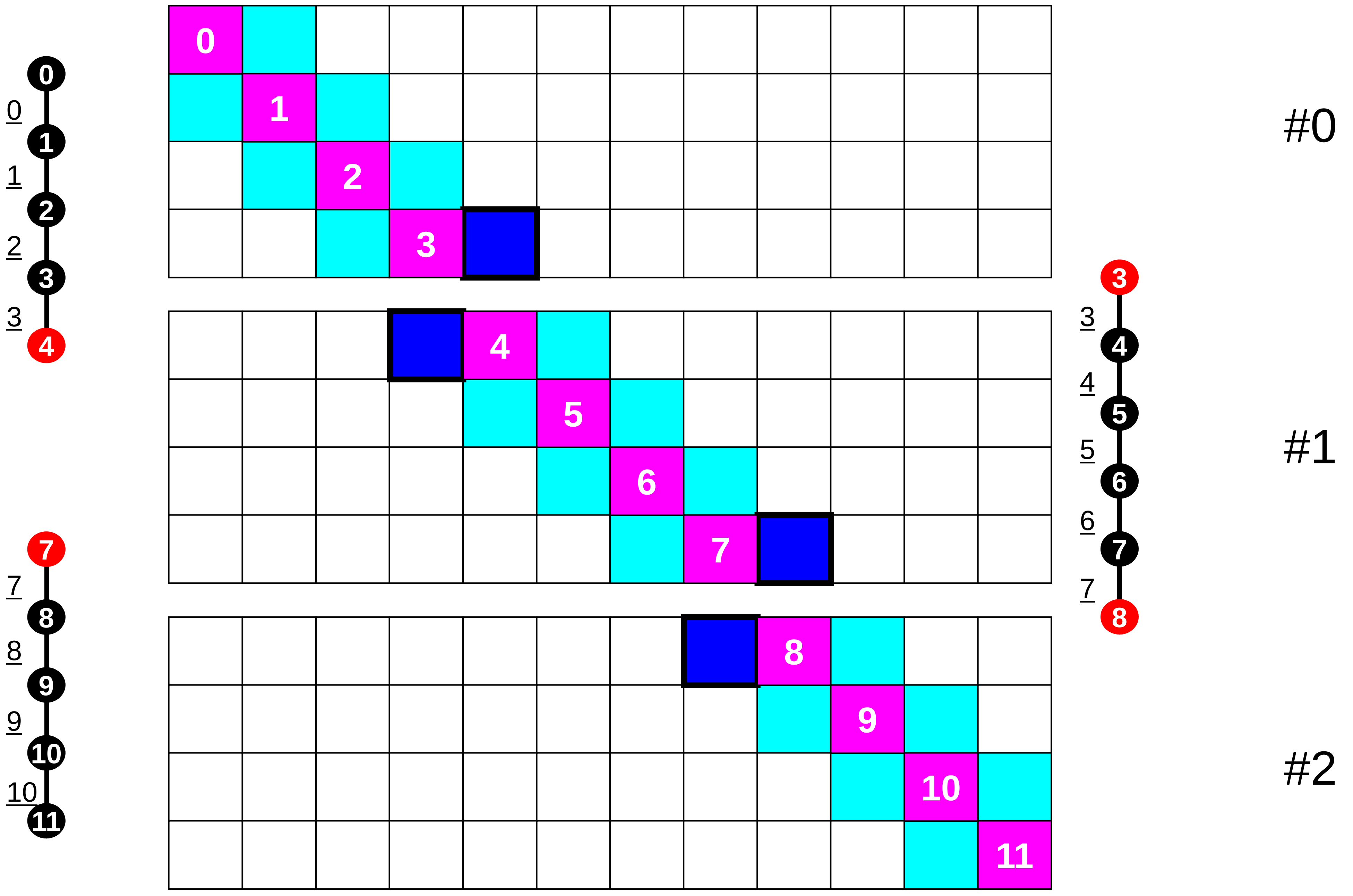
# “Internal Nodes” should be balanced



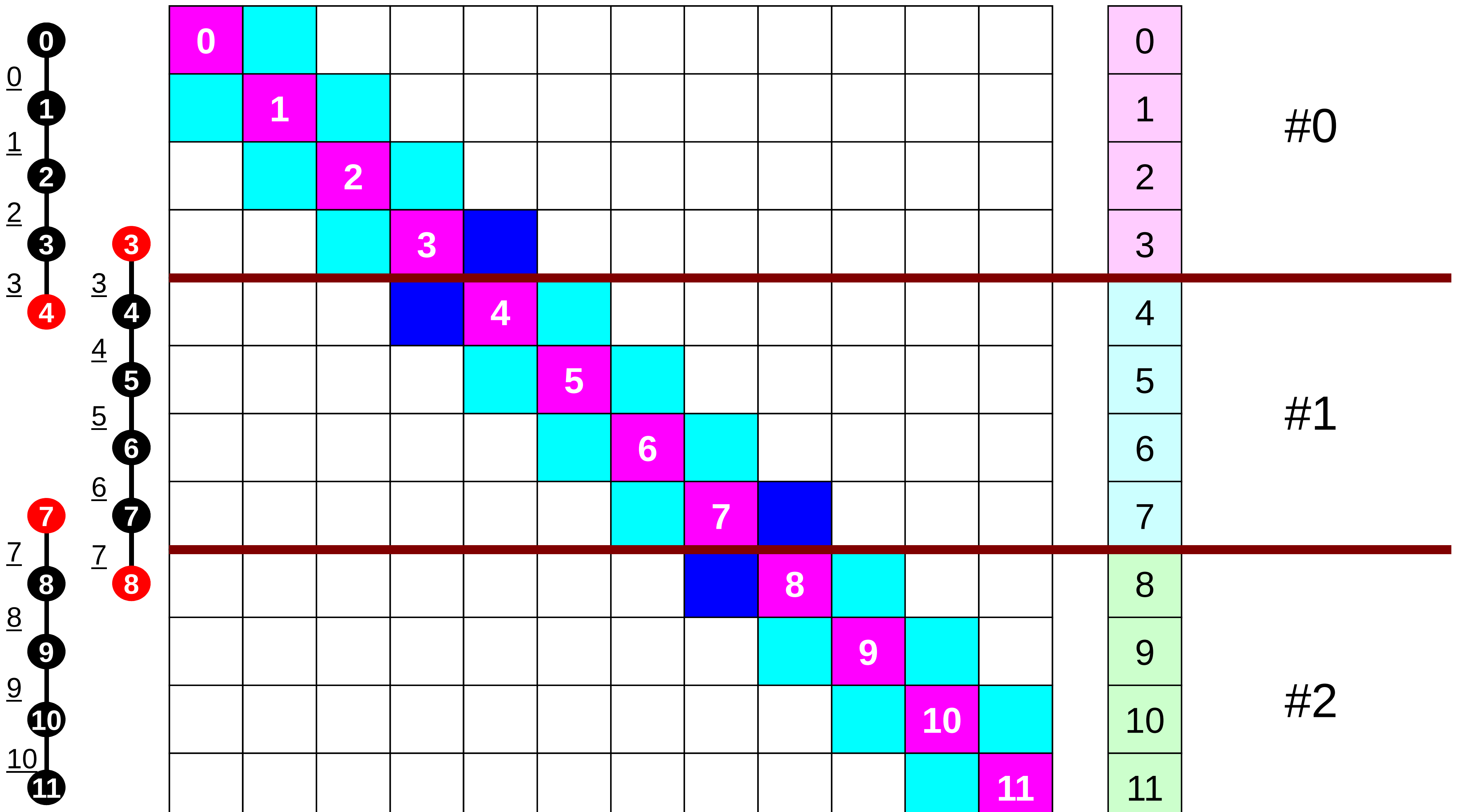
Matrices are incomplete !



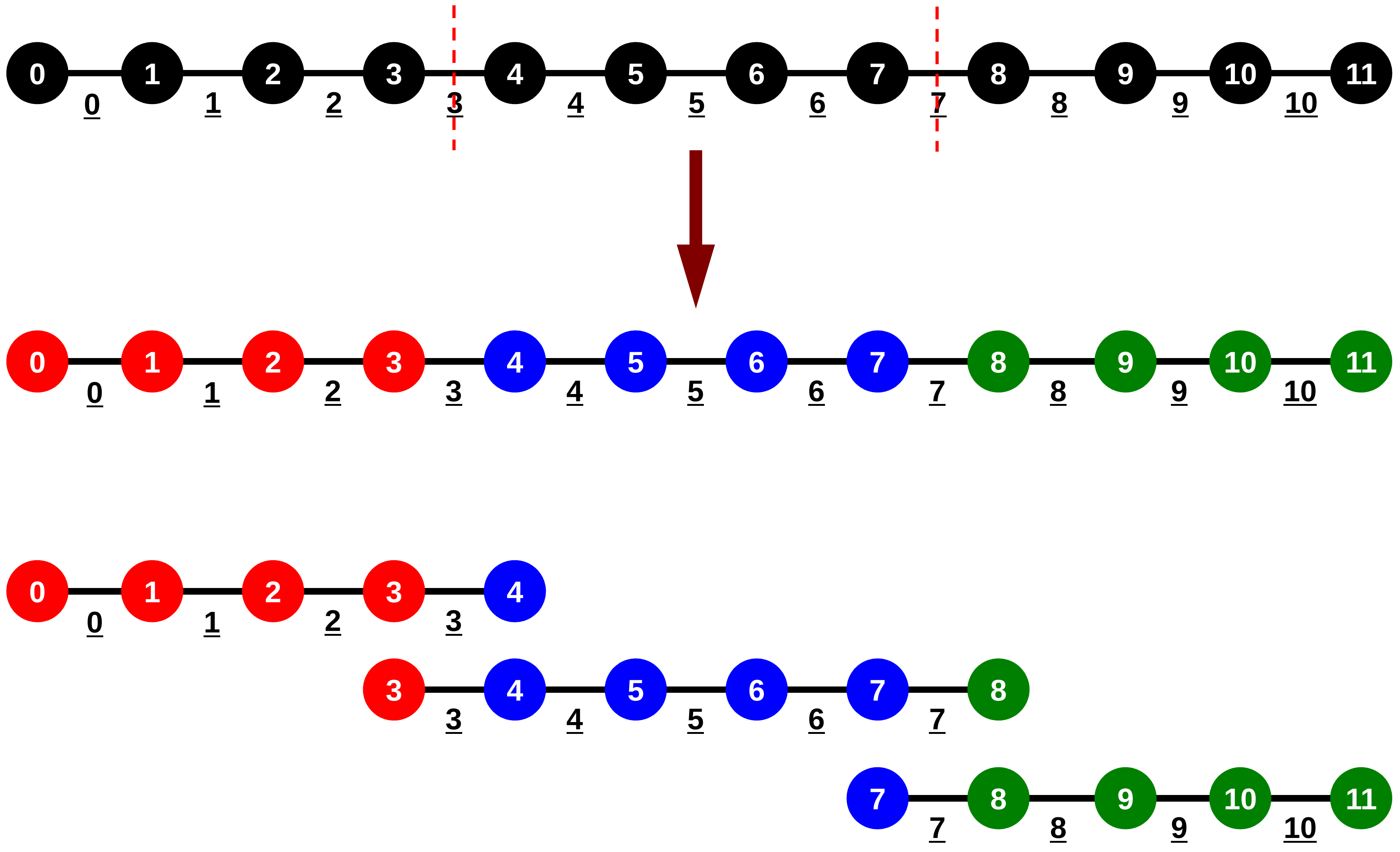
Connected Elements + External Nodes



1D FEM: 12 nodes/11 elem's/3 domains

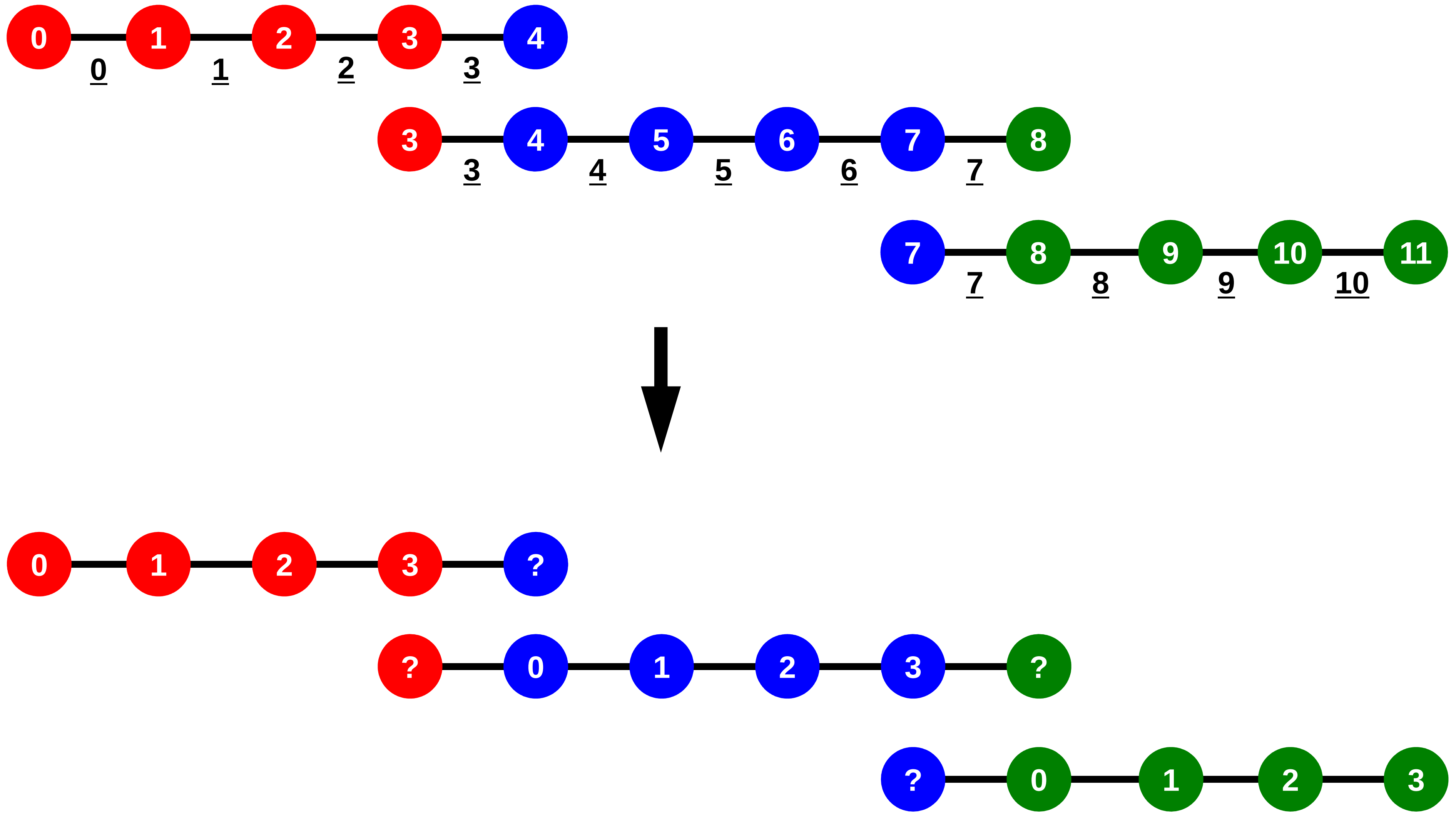


1D FEM: 12 nodes/11 elem's/3 domains



Local Numbering for SPMD

Numbering of internal nodes is 1-N (0-N-1), same operations in serial program can be applied. How about numbering of external nodes ?

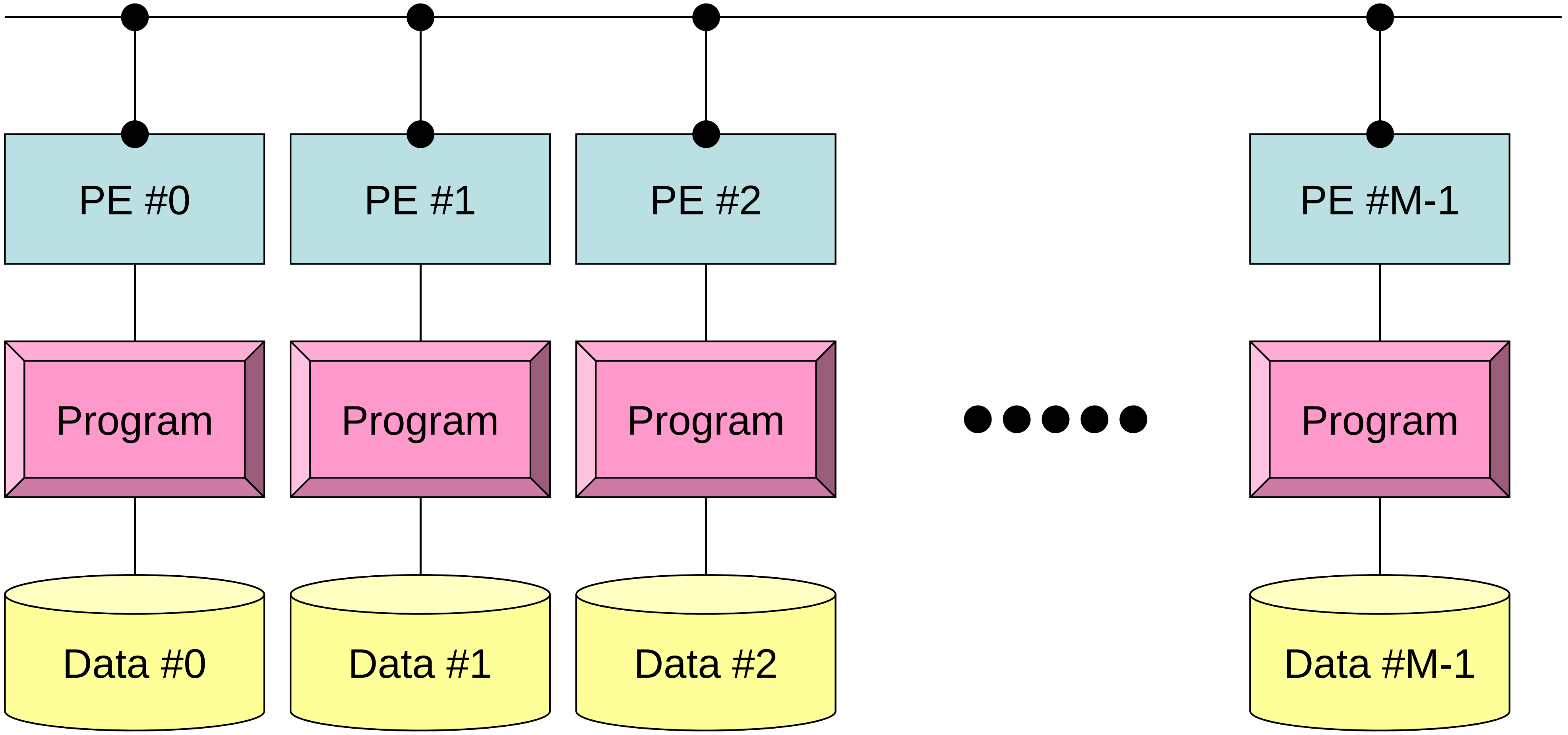


SPMD:

Single Program Multiple Data

PE: Processing Element
Processor, Domain, Process

```
mpirun -np M <Program>
```



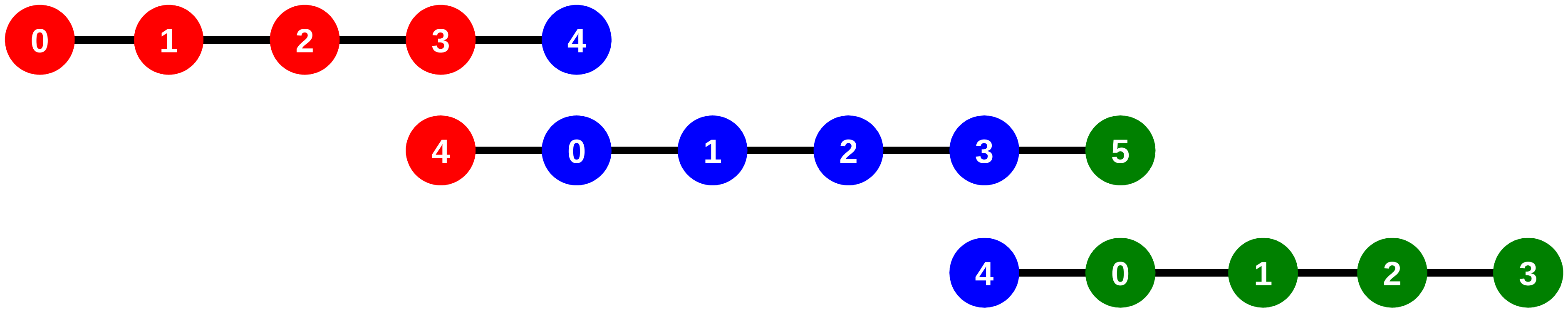
Each process does same operation for different data

Large-scale data is decomposed, and each part is computed by each process

It is ideal that parallel program is not different from serial one except communication.

Local Numbering for SPMD

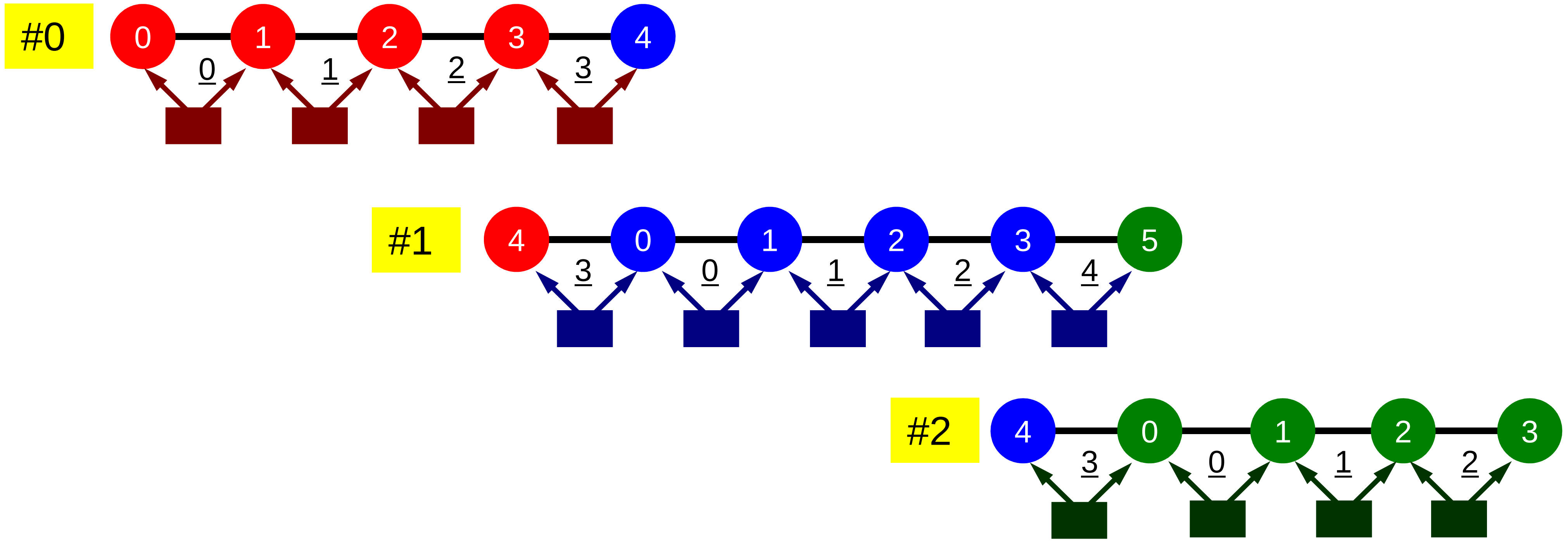
Numbering of external nodes: N+1, N+2 (N,N+1)



1D FEM: 12 nodes/11 elem’s/3 domains

Integration on each element, element matrix -> global matrix

Operations can be done by info. of internal/external nodes and elements which include these nodes



Finite Element Procedures

- Initialization
  - Control Data
  - Node, Connectivity of Elements (N: Node#, NE: Elem#)
  - Initialization of Arrays (Global/Element Matrices)
  - Element-Global Matrix Mapping (Index, Item)
- Generation of Matrix
  - Element-by-Element Operations (do icel= 1, NE)
    - Element matrices
    - Accumulation to global matrix
  - Boundary Conditions
- Linear Solver
  - Conjugate Gradient Method

Preconditioned CG Solver

```
Compute $r^{(0)}= b-[A]x^{(0)}$
for i= 1, 2, …
    solve $[M]z^{(i-1)}= r^{(i-1)}$
    $\rho_{i-1}= r^{(i-1)} z^{(i-1)}$
    if i=1
      $p^{(1)}= z^{(0)}$
     else
      $\beta_{i-1}= \rho_{i-1}/\rho_{i-2}$
      $p^{(i)}= z^{(i-1)} + \beta_{i-1} p^{(i-1)}$
    endif
    $q^{(i)}= [A]p^{(i)}$
    $\alpha_i = \rho_{i-1}/p^{(i)}q^{(i)}$
    $x^{(i)}= x^{(i-1)} + \alpha_i p^{(i)}$
    $r^{(i)}= r^{(i-1)} - \alpha_i q^{(i)}$
    check convergence $|r|$
end
```

- Preconditioning
  - Diagonal Scaling/Point Jacobi
- Parallel operations are required in
  - Dot Products
  - Mat-Vec. Multiplication
    - SpMV: Sparse Mat-Vec. Mult.

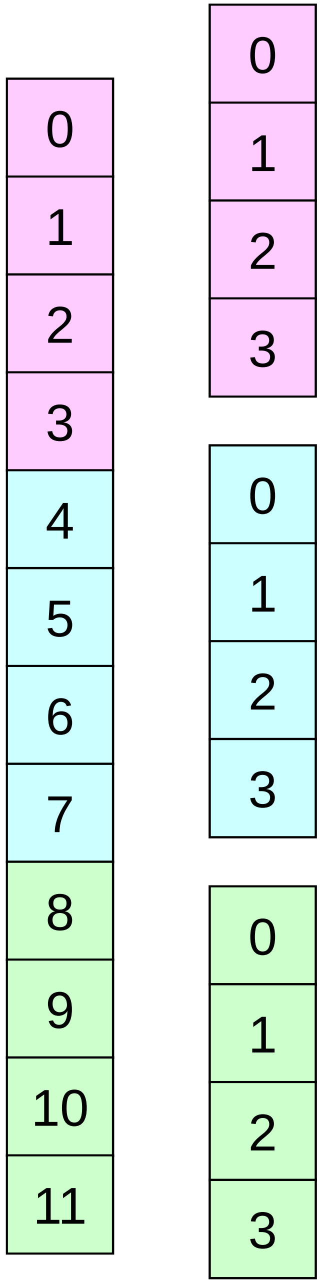
$$[M]=\begin{bmatrix} D_1 & 0 & \dots & 0 & 0 \\ 0 & D_2 & & 0 & 0 \\ \dots & & \dots & & \dots \\ 0 & 0 & & D_{N-1} & 0 \\ 0 & 0 & \dots & 0 & D_N \end{bmatrix}$$

Preconditioning, DAXPY

Local Operations by Only Internal Points: Parallel Processing is possible

```
/*
//-- {z}= [Minv]{r}
*/
    for(i=0;i<N;i++){
        W[Z][i] = W[DD][i] * W[R][i];
    }
```

```
/*
//-- {x}= {x} + ALPHA*{p}       DAXPY: double a{x} plus {y}
//    {r}= {r} - ALPHA*{q}
*/
  for(i=0;i<N;i++){
      U[i]    += Alpha * W[P][i];
      W[R][i] -= Alpha * W[Q][i];
  }
```

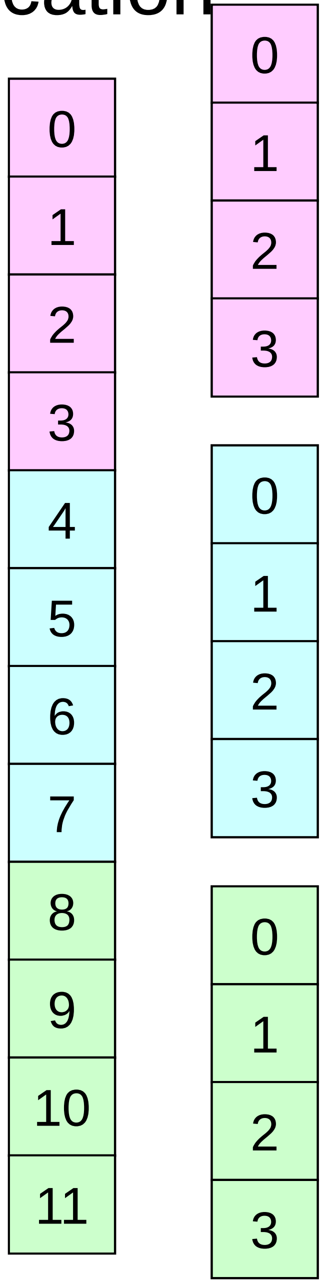


Dot Products

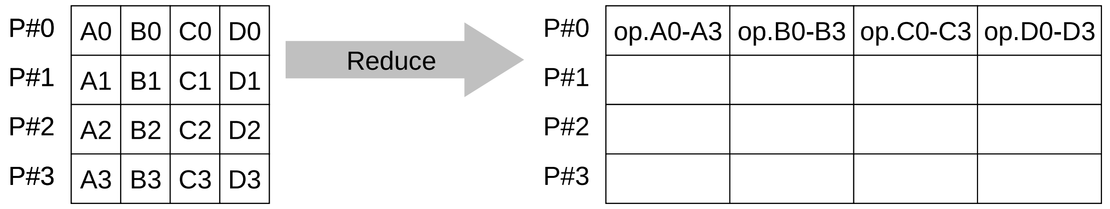
Global Summation needed: Communication ?

```
/*
//-- ALPHA= RHO / {p}{q}
*/
  C1 = 0.0;
  for(i=0;i<N;i++){
      C1 += W[P][i] * W[Q][i];
  }

  Alpha = Rho / C1;
```



MPI_Reduce



- Reduces values on all processes to a single value
  - Summation, Product, Max, Min etc.

- **`MPI_Reduce (sendbuf,recvbuf,count,datatype,op,root,comm)`**

– **sendbuf**	choice	I	starting address of send buffer
– **recvbuf**	choice	O	starting address receive buffer
			type is defined by "**datatype**"
– **count**	int	I	number of elements in send/receive buffer
– **datatype**	MPI_Datatype	I	data type of elements of send/recive buffer

```
FORTRAN  MPI_INTEGER, MPI_REAL, MPI_DOUBLE_PRECISION, MPI_CHARACTER etc.
C        MPI_INT, MPI_FLOAT, MPI_DOUBLE, MPI_CHAR etc
```

– **op**	MPI_Op	I	reduce operation

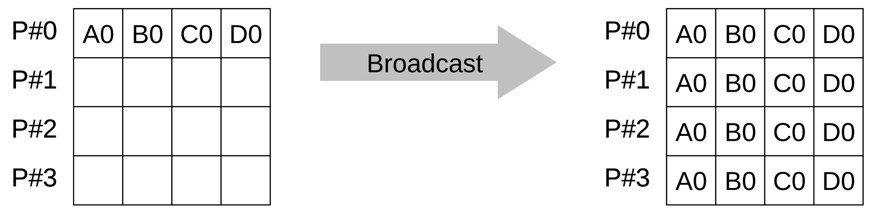
```
MPI_MAX, MPI_MIN, MPI_SUM, MPI_PROD, MPI_LAND, MPI_BAND etc
```

Users can define operations by **MPI_OP_CREATE**

– **root**	int	I	rank of root process
– **comm**	MPI_Comm	I	communicator

C

MPI_Bcast



- Broadcasts a message from the process with rank "root" to all other processes of the communicator

- **`MPI_Bcast (buffer,count,datatype,root,comm)`**

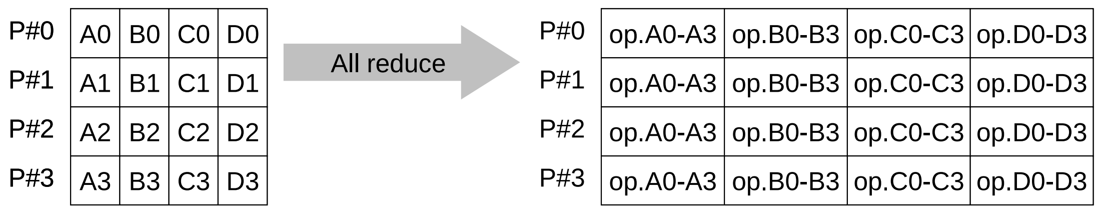
– **<u>`buffer`</u>**	choice	`I/O`	starting address of buffer type is defined by "**<u>`datatype`</u>**"
– **<u>`count`</u>**	int	`I`	number of elements in send/recv buffer
– **<u>`datatype`</u>**	MPI_Datatype	`I`	data type of elements of send/recv buffer

```
FORTRAN  MPI_INTEGER, MPI_REAL, MPI_DOUBLE_PRECISION, MPI_CHARACTER etc.
C        MPI_INT, MPI_FLOAT, MPI_DOUBLE, MPI_CHAR etc.
```

– **<u>`root`</u>**	int	`I`	rank of root process
– **<u>`comm`</u>**	MPI_Comm	`I`	communicator

C

MPI_Allreduce


P#0 A0 B0 C0 D0
P#1 A1 B1 C1 D1
P#2 A2 B2 C2 D2
P#3 A3 B3 C3 D3
All reduce
P#0 op.A0-A3 op.B0-B3 op.C0-C3 op.D0-D3
P#1 op.A0-A3 op.B0-B3 op.C0-C3 op.D0-D3
P#2 op.A0-A3 op.B0-B3 op.C0-C3 op.D0-D3
P#3 op.A0-A3 op.B0-B3 op.C0-C3 op.D0-D3

- MPI_Reduce + MPI_Bcast
- Summation (of dot products) and MAX/MIN values are likely to utilized in each process

- **`call MPI_Allreduce (sendbuf,recvbuf,count,datatype,op, comm)`**

– **<u>`sendbuf`</u>**	choice	I	starting address of send buffer
– **<u>`recvbuf`</u>**	choice	O	starting address receive buffer type is defined by "**<u>`datatype`</u>**"
– **<u>`count`</u>**	int	I	number of elements in send/recv buffer
– **<u>`datatype`</u>**	MPI_Datatype	I	data type of elements of send/recv buffer
– **<u>`op`</u>**	MPI_Op	I	reduce operation
– **<u>`comm`</u>**	MPI_Comm	I	communicator

C

C

“op” of MPI_Reduce/Allreduce

```
MPI_Reduce
 (sendbuf,recvbuf,count,datatype,op,root,comm)
```

• **`MPI_MAX`**, **`MPI_MIN`**	Max, Min
• **`MPI_SUM`**, **`MPI_PROD`**	Summation, Product
• **`MPI_LAND`**	Logical AND

Preconditioned CG Solver

```
Compute $r^{(0)}= b-[A]x^{(0)}$
for i= 1, 2, …
    solve $[M]z^{(i-1)}= r^{(i-1)}$
    $\rho_{i-1}= r^{(i-1)} z^{(i-1)}$
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     else
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    endif
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    $\alpha_i = \rho_{i-1}/p^{(i)}q^{(i)}$
    $x^{(i)}= x^{(i-1)} + \alpha_i p^{(i)}$
    $r^{(i)}= r^{(i-1)} - \alpha_i q^{(i)}$
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end
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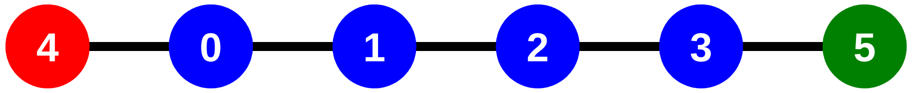
- Preconditioning
  - Diagonal Scaling/Point Jacobi
- Parallel operations are required in
  - Dot Products
  - Mat-Vec. Multiplication
    - SpMV: Sparse Mat-Vec. Mult.

$$[M]=\begin{bmatrix} D_1 & 0 & \ldots & 0 & 0 \\ 0 & D_2 & & 0 & 0 \\ \ldots & & \ldots & & \ldots \\ 0 & 0 & & D_{N-1} & 0 \\ 0 & 0 & \ldots & 0 & D_N \end{bmatrix}$$

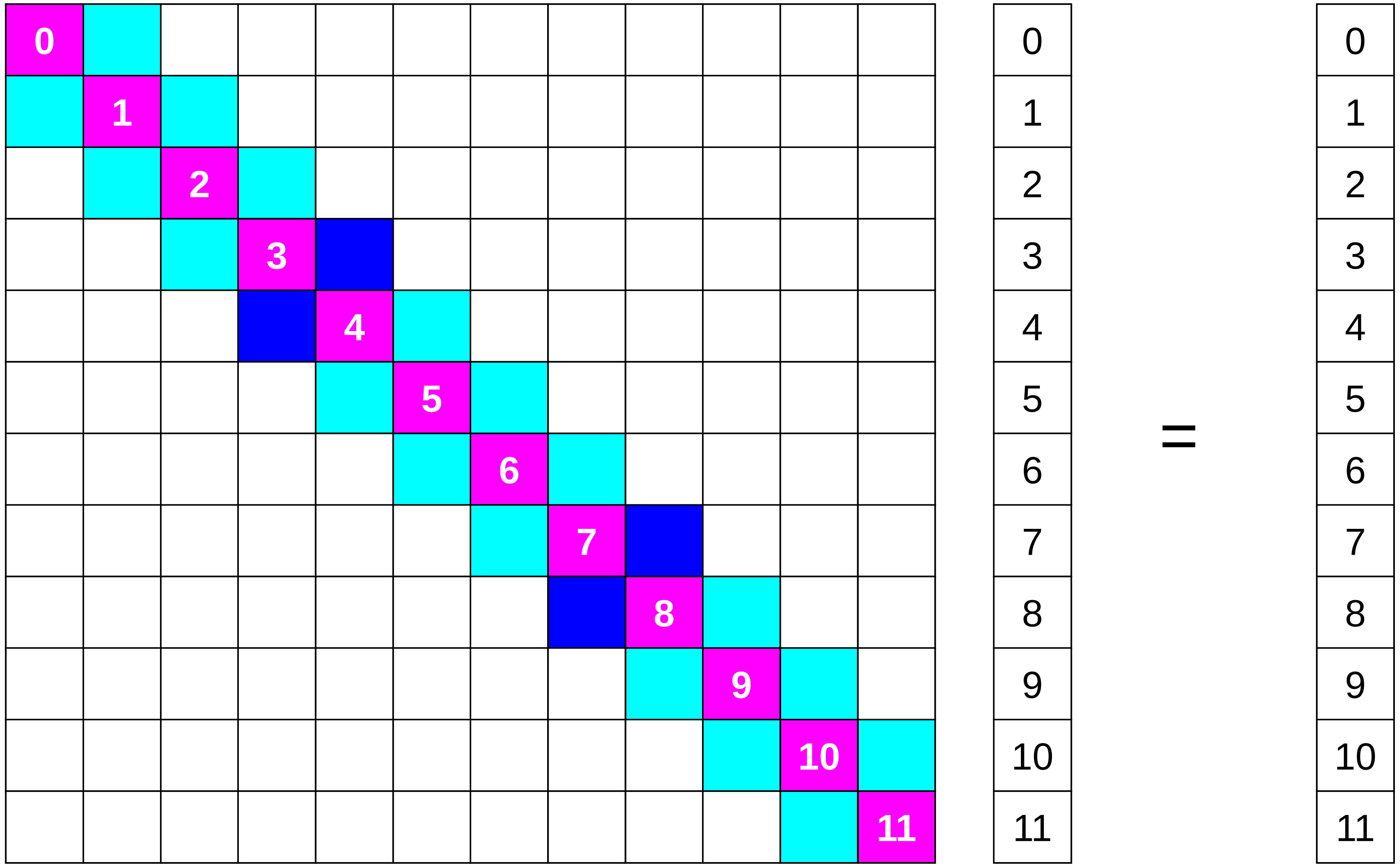
Matrix-Vector Products

Values at External Points: P-to-P Communication

```
/*
//-- {q}= [A]{p}
*/
   for(i=0;i<N;i++){
         W[Q][i] = Diag[i] * W[P][i];
         for(j=Index[i];j<Index[i+1];j++){
                W[Q][i] += AMat[j]*W[P][Item[j]];
                }
         }
```



Mat-Vec Products: Local Op. Possible



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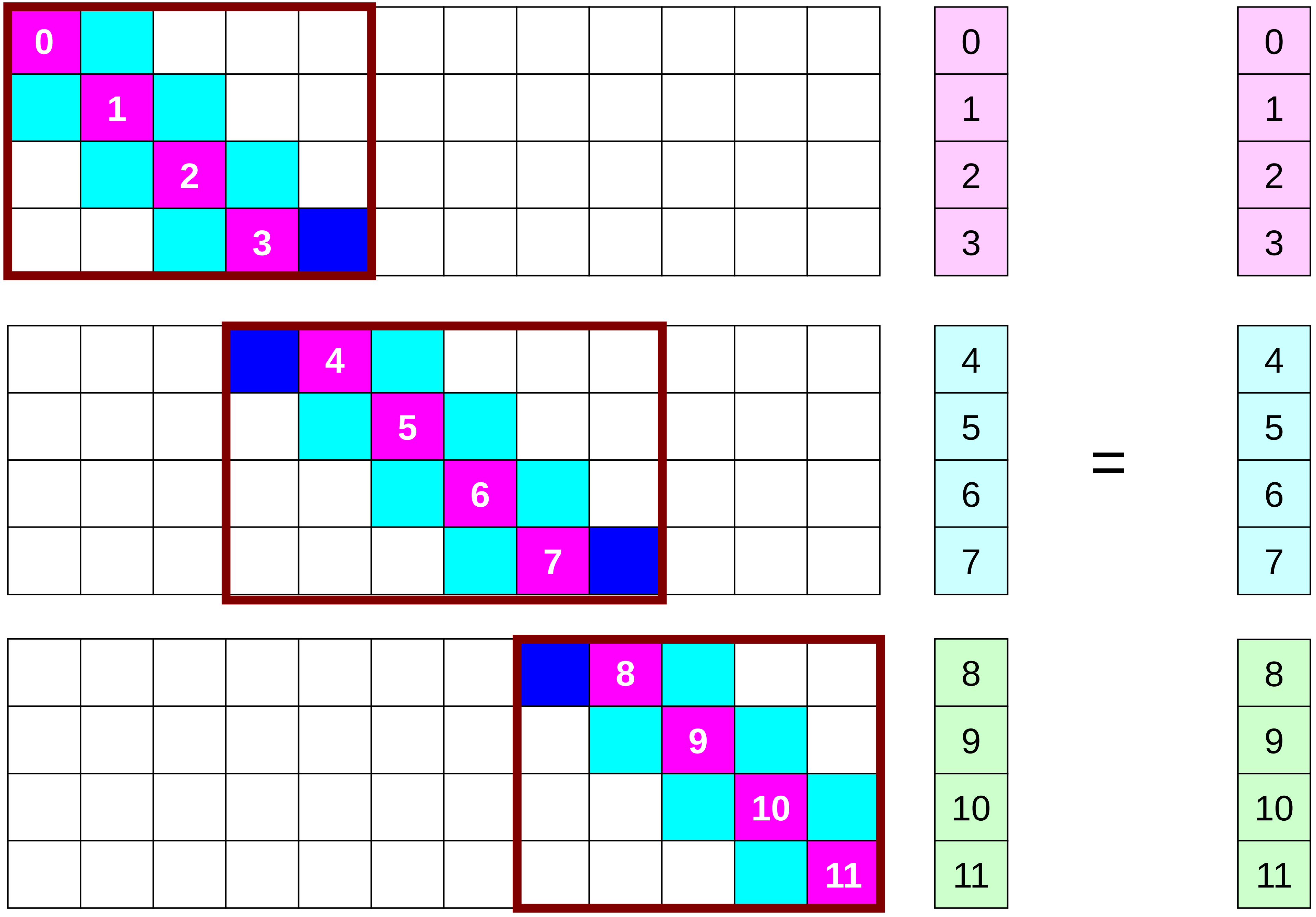
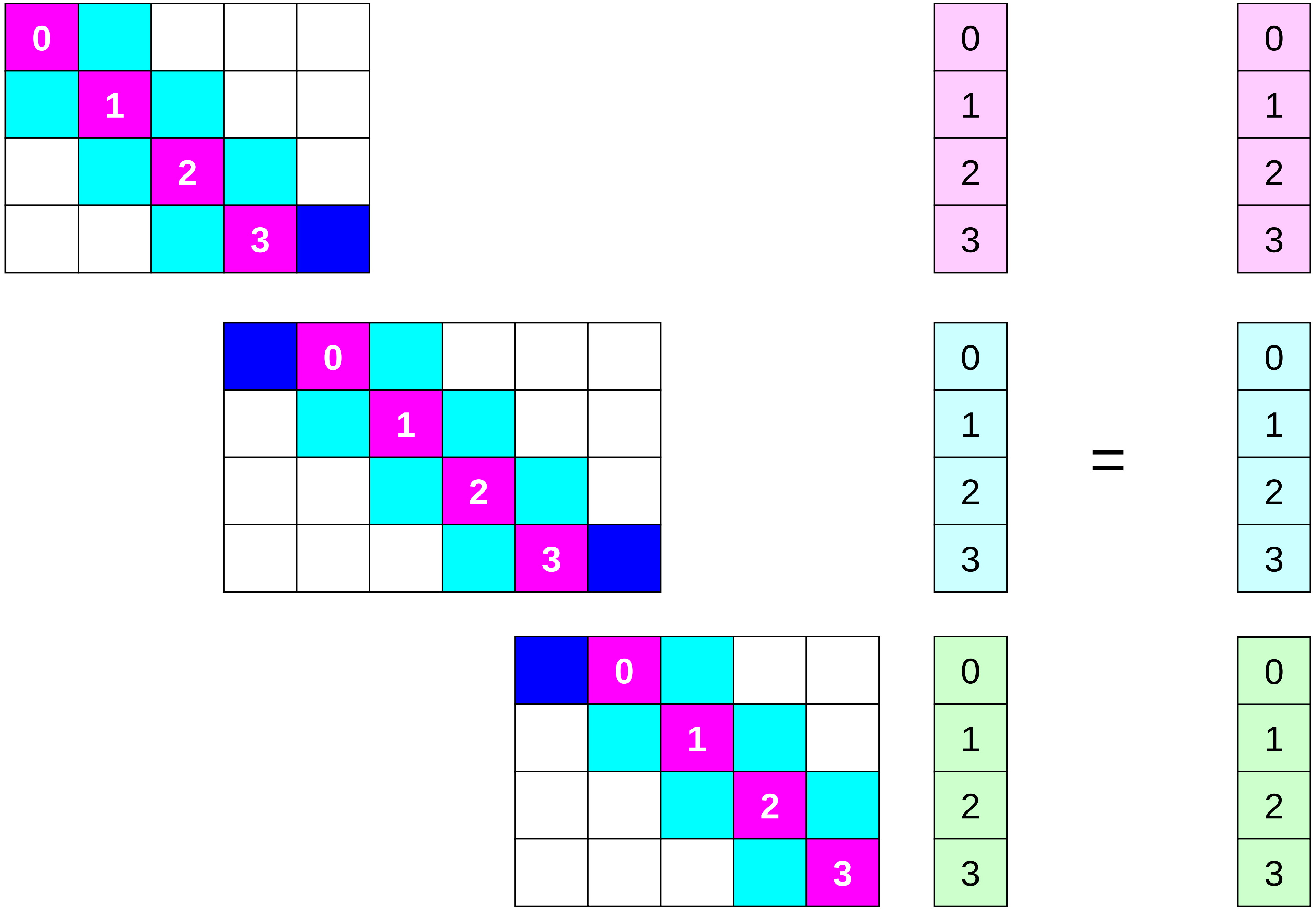
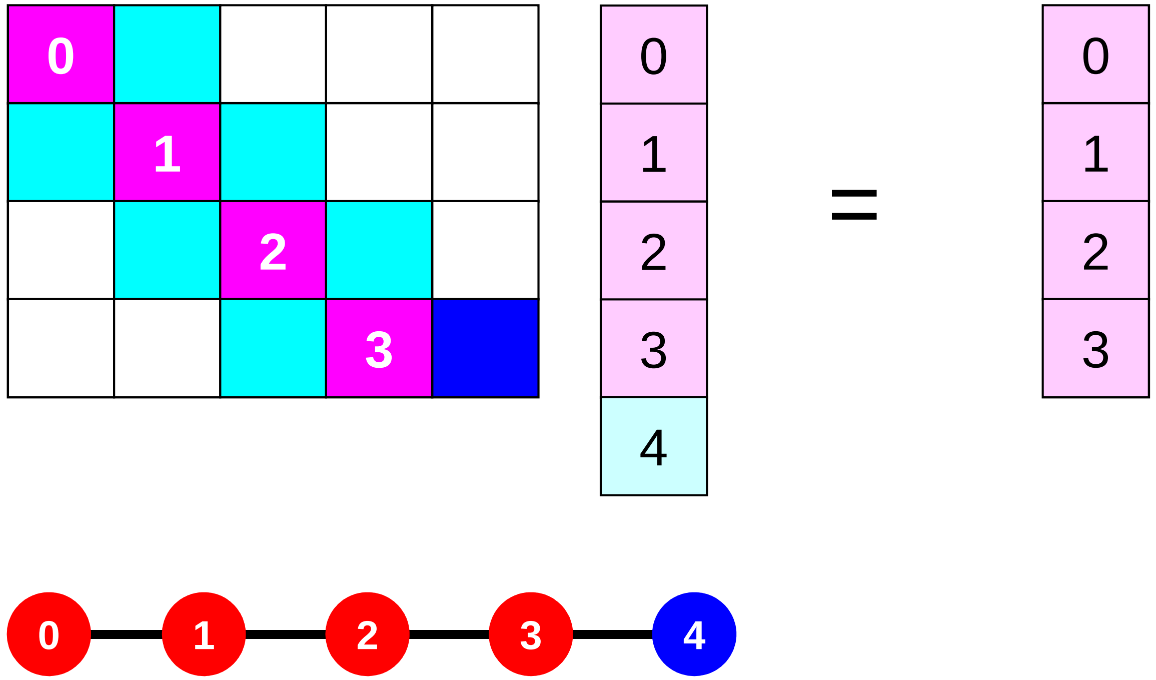


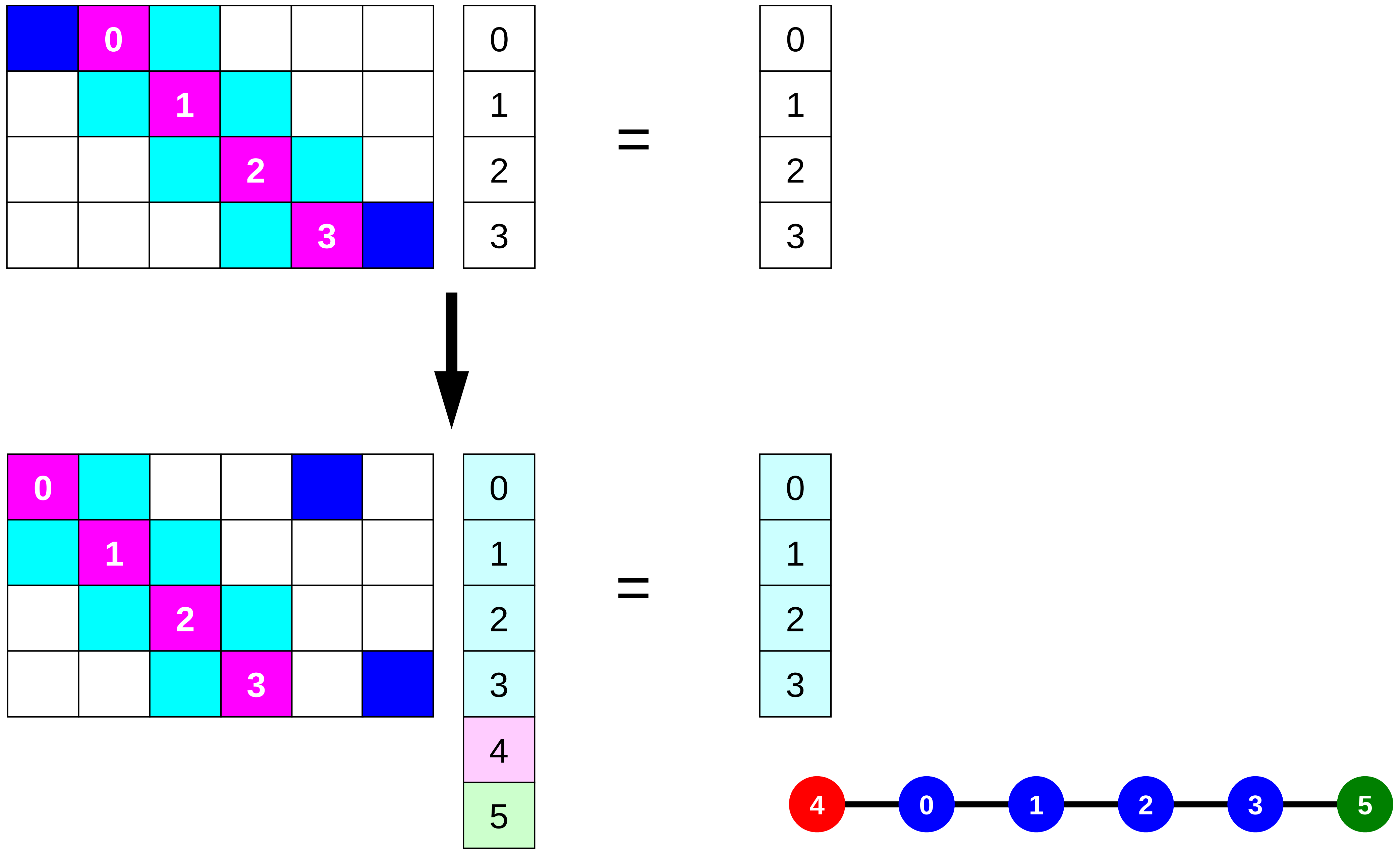
Mat-Vec Products: Local Op. Possible



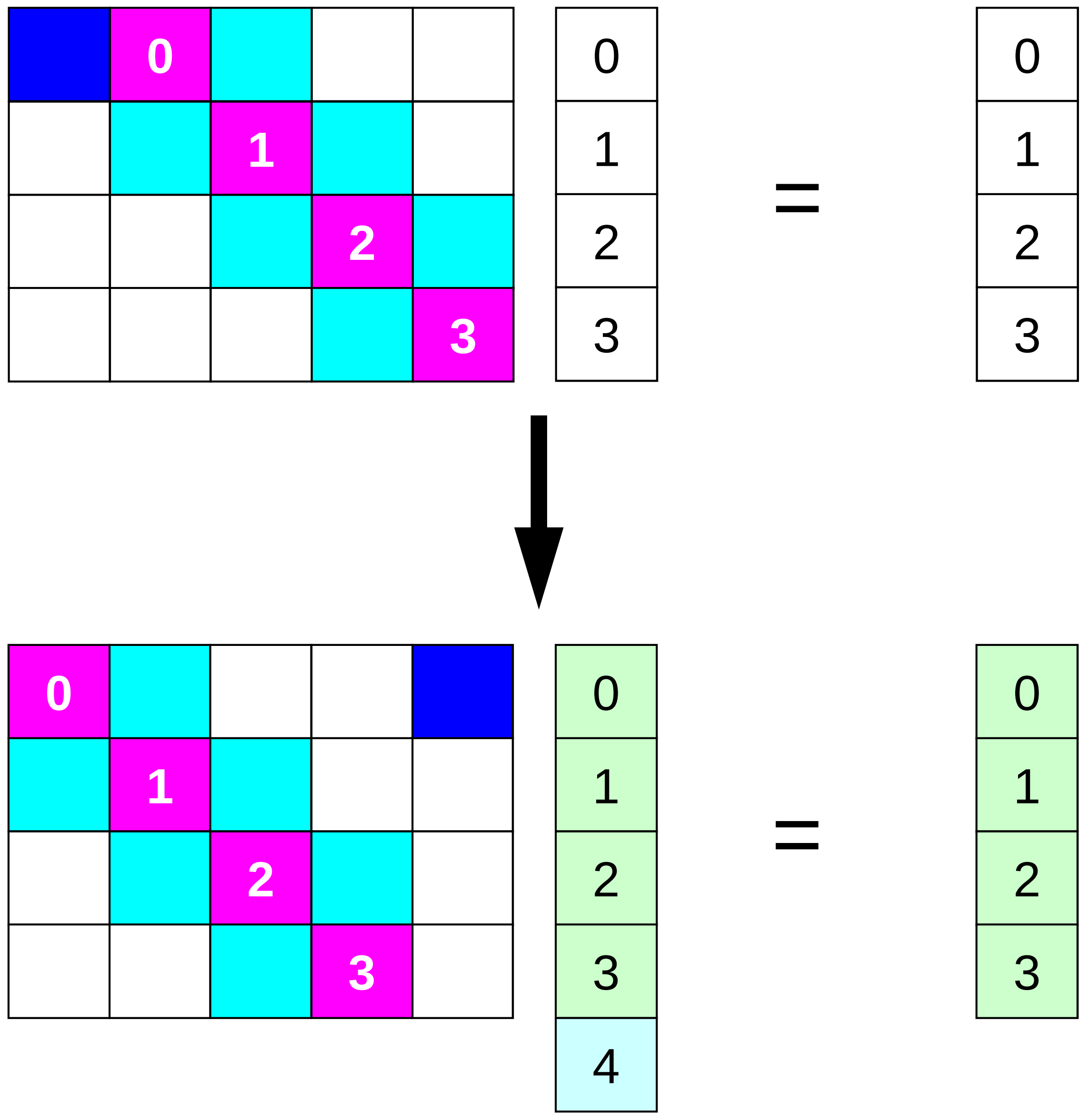
Mat-Vec Products: Local Op. #0

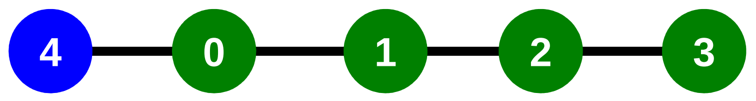


Mat-Vec Products: Local Op. #1

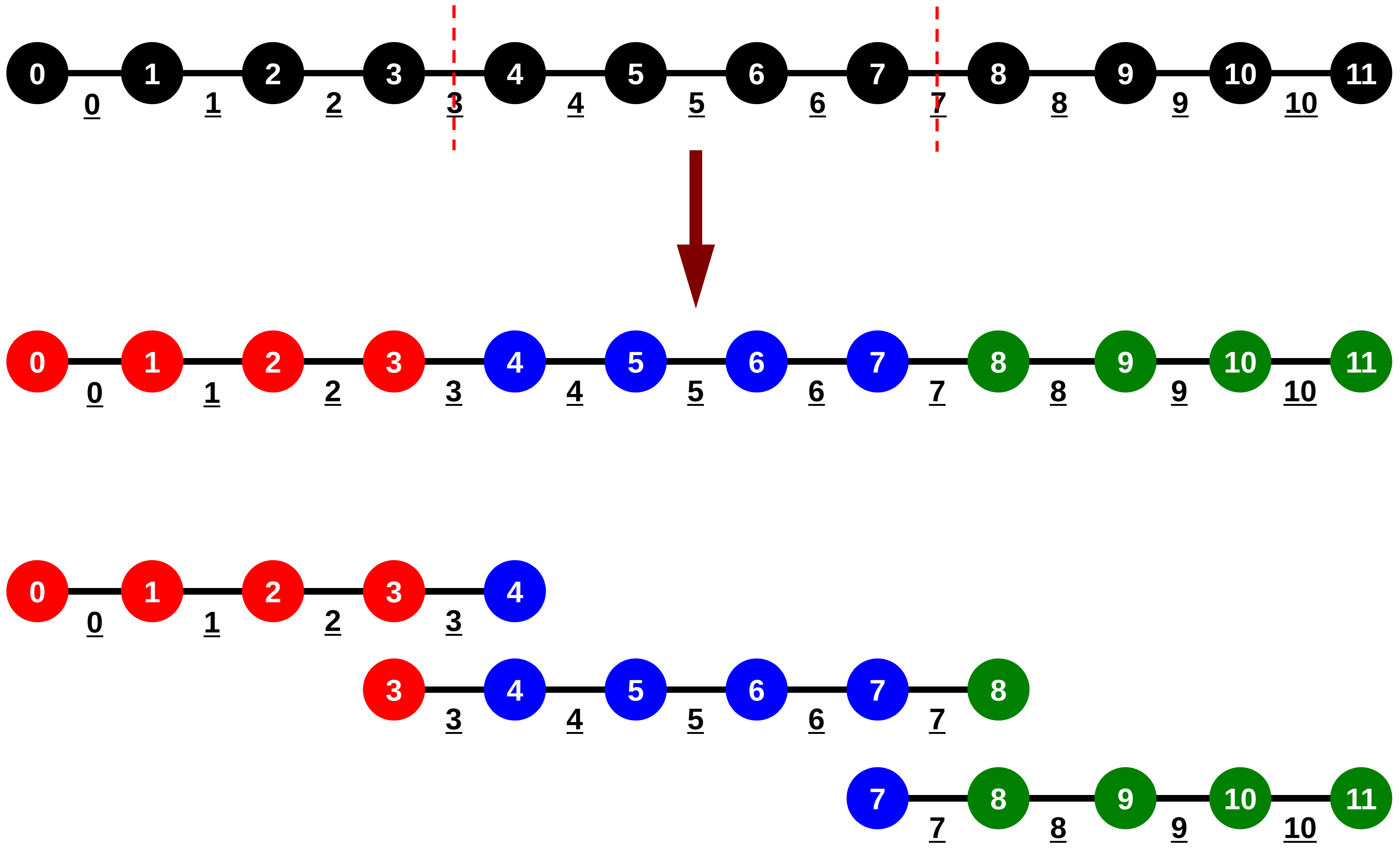


Mat-Vec Products: Local Op. #2



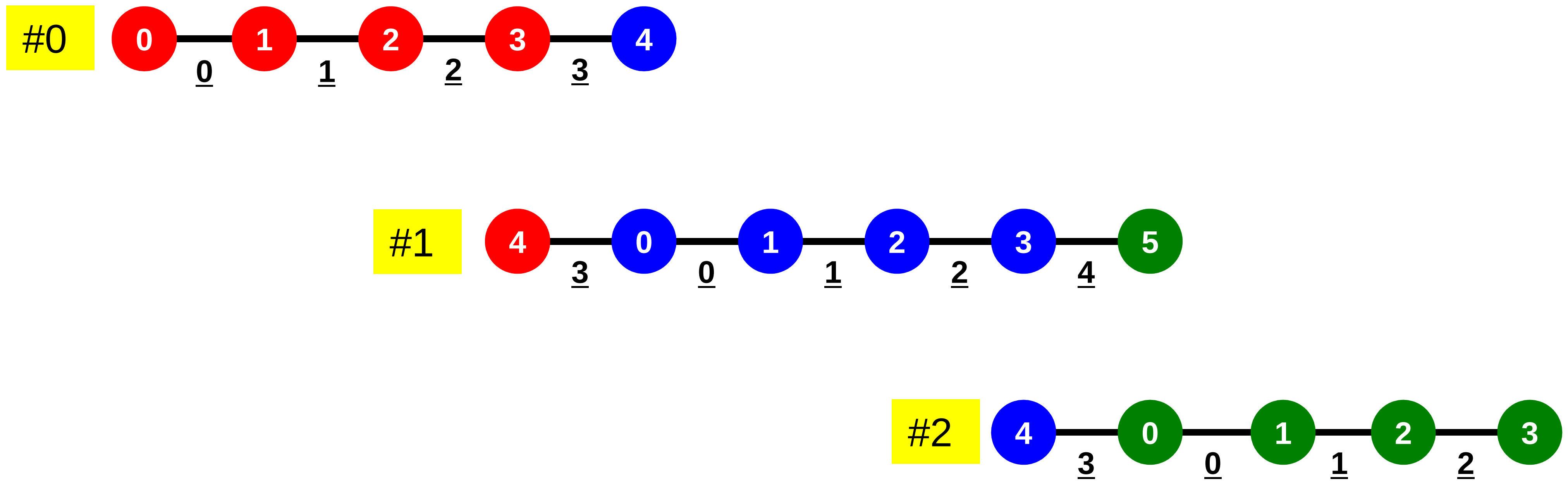


1D FEM: 12 nodes/11 elem's/3 domains



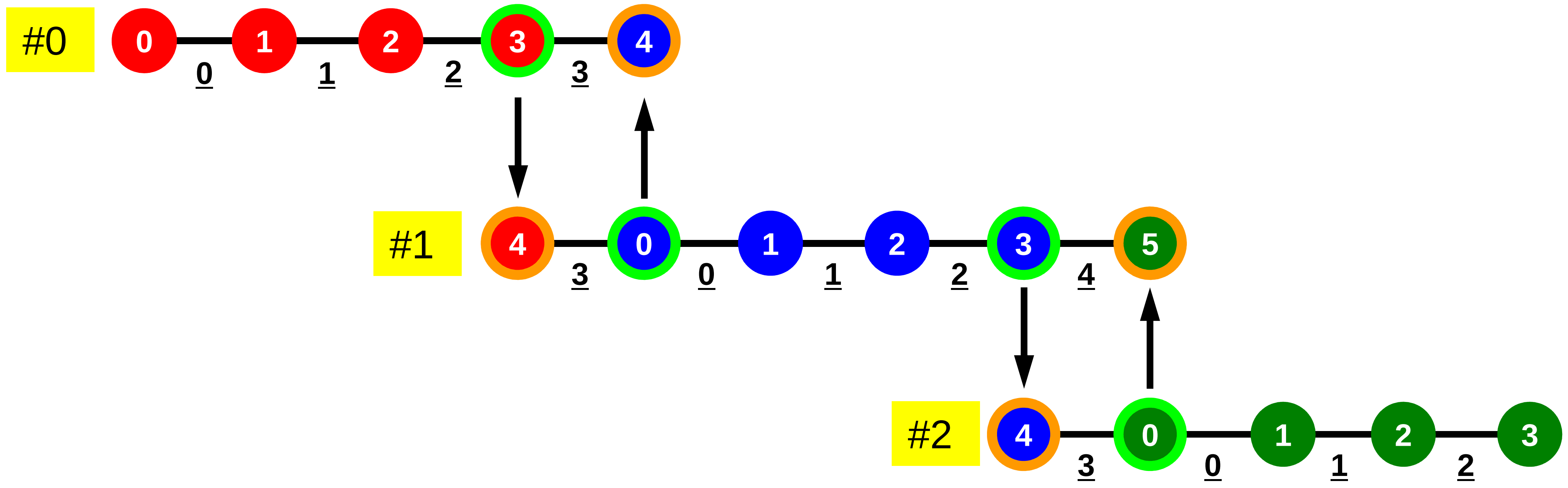
1D FEM: 12 nodes/11 elem’s/3 domains

Local ID: Starting from 0 for node and elem at each domain



1D FEM: 12 nodes/11 elem’s/3 domains

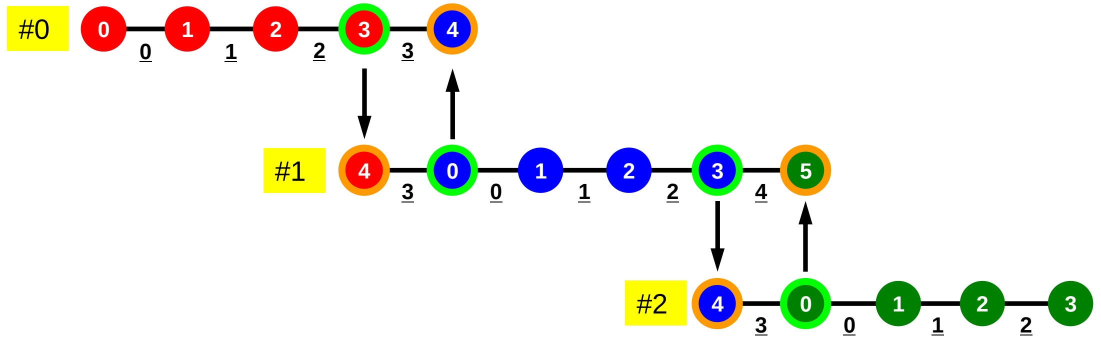
Internal/External Nodes



What is Point-to-Point Communication ?

- Collective Communication
  - MPI_Reduce, MPI_Scatter/Gather etc.
  - Communications with all processes in the communicator
  - Application Area
    - BEM, Spectral Method, MD: global interactions are considered
    - Dot products, MAX/MIN: Global Summation & Comparison

- Point-to-Point
  - MPI_Send, MPI_Recv
  - Communication with limited processes
    - Neighbors
  - Application Area
    - FEM, FDM: Localized Method

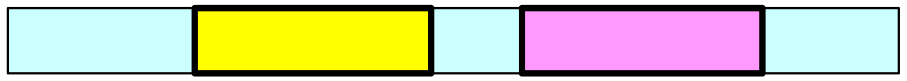


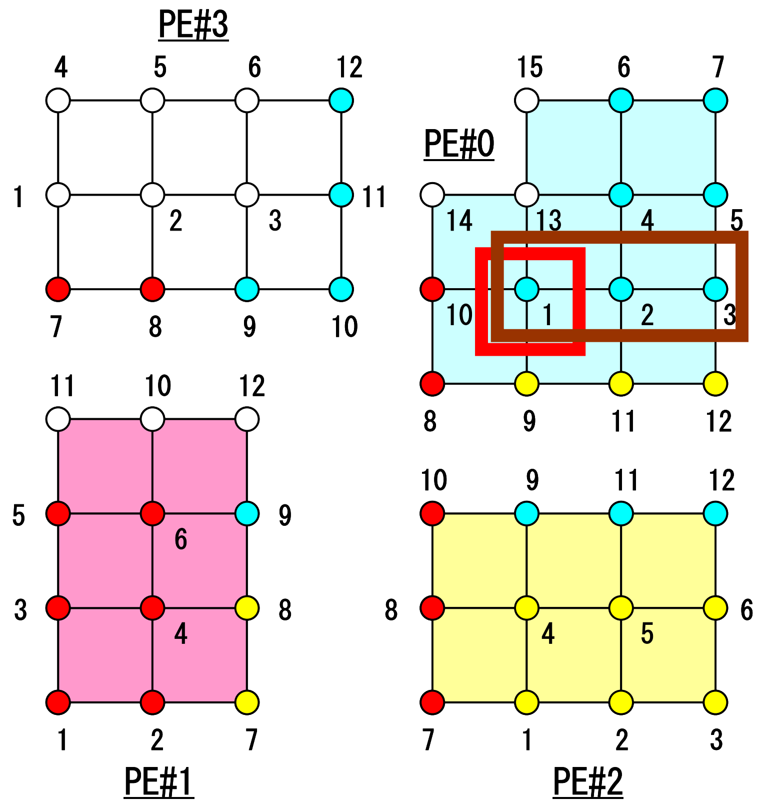
SEND: sending from boundary nodes

Send continuous data to send buffer of neighbors

- **`MPI_Isend`**
  `(sendbuf,count,datatype,dest,tag,comm,request)`

– **sendbuf**	choice	I	starting address of sending buffer
– **count**	I	I	number of elements sent to each process
– **datatype**	I	I	data type of elements of sending buffer
– **dest**	I	I	rank of destination





C

MPI_Isend

- Begins a non-blocking send
  - Send the contents of sending buffer (starting from **`sendbuf`**, number of messages: **`count`**) to **`dest`** with **`tag`** .
  - Contents of sending buffer cannot be modified before calling corresponding **`MPI_Waitall`**.

- **`MPI_Isend (sendbuf,count,datatype,dest,tag,comm,request)`**

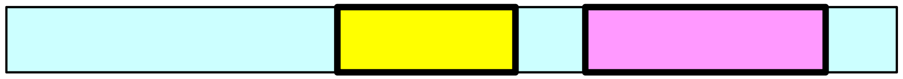
– **`sendbuf`**	choice	I	starting address of sending buffer
– **`count`**	int	I	number of elements in sending buffer
– **`datatype`**	MPI_Datatype	I	datatype of each sending buffer element
– **`dest`**	int	I	rank of destination
– **`tag`**	int	I	message tag This integer can be used by the application to distinguish messages. Communication occurs if `tag's` of `MPI_Isend` and `MPI_Irecv` are matched. Usually tag is set to be "0" (in this class),
– **`comm`**	MPI_Comm	I	communicator
– **`request`**	MPI_Request	O	communication request array used in `MPI_Waitall`

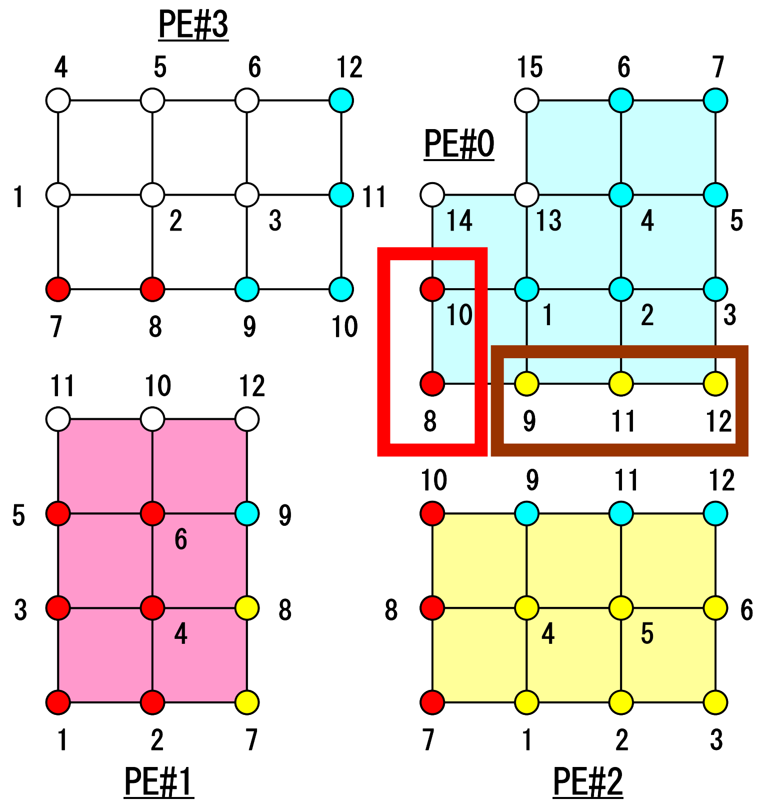
RECV: receiving to <u>external</u> nodes

Recv. continuous data to recv. buffer from neighbors

- **`MPI_Irecv`**
  `(recvbuf,count,datatype,dest,tag,comm,request)`

–	**`recvbuf`**	choice	`I`	starting address of receiving buffer
–	**`count`**	`I`	`I`	number of elements in receiving buffer
–	**`datatype`**	`I`	`I`	data type of elements of receiving buffer
–	**`source`**	`I`	`I`	rank of source





C

MPI_Irecv

- Begins a non-blocking receive
  - Receiving the contents of receiving buffer (starting from **`recvbuf`**, number of messages: **`count`**) from **`source`** with **`tag`** .
  - Contents of receiving buffer cannot be used before calling corresponding **`MPI_Waitall`**.

- **`MPI_Irecv (recvbuf,count,datatype,source,tag,comm,request)`**

– **recvbuf**	choice	I	starting address of receiving buffer
– **count**	int	I	number of elements in receiving buffer
– **datatype**	MPI_Datatype	I	datatype of each receiving buffer element
– **source**	int	I	rank of source
– **tag**	int	I	message tag This integer can be used by the application to distinguish messages. Communication occurs if `tag's` of `MPI_Isend` and `MPI_Irecv` are matched. Usually tag is set to be "0" (in this class),
– **comm**	MPI_Comm	I	communicator
– **request**	MPI_Request	O	communication request array used in `MPI_Waitall`

MPI_Waitall

C

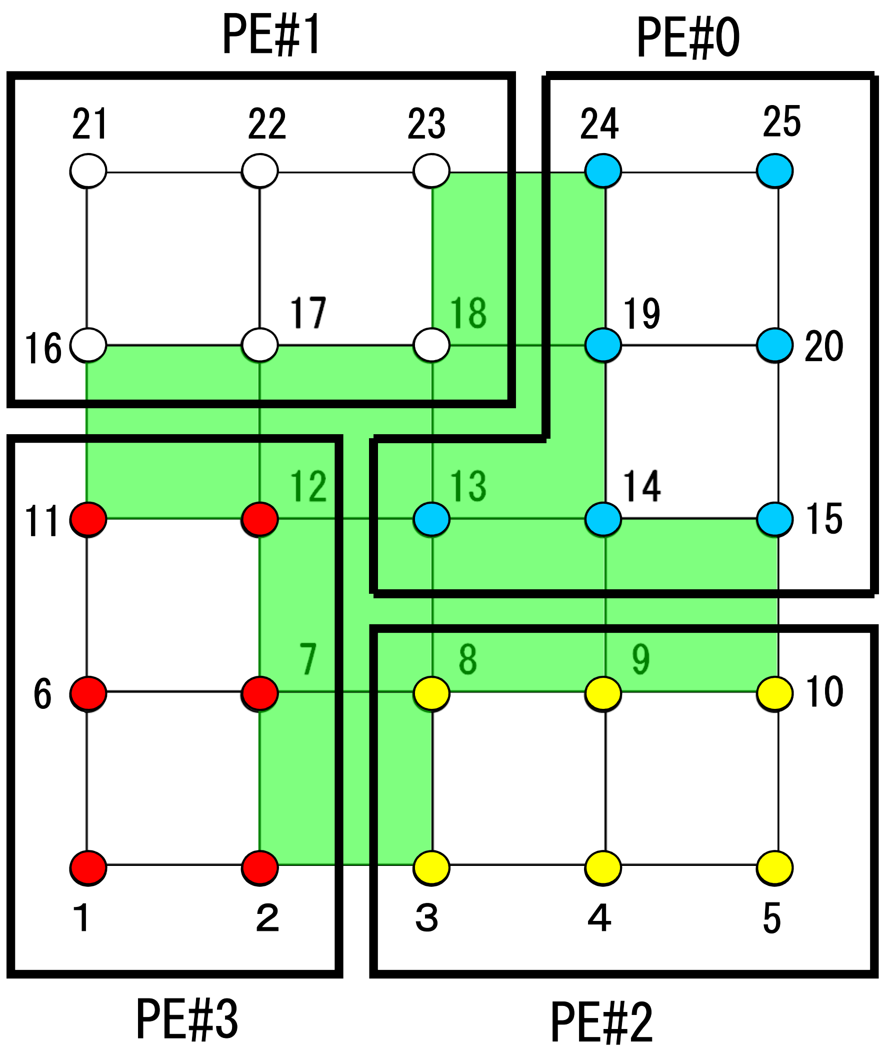
- **MPI_Waitall** blocks until all comm's, associated with **request** in the array, complete. It is used for synchronizing **MPI_Isend** and **MPI_Irecv** in this class.
- At sending phase, contents of sending buffer cannot be modified before calling corresponding **MPI_Waitall**. At receiving phase, contents of receiving buffer cannot be used before calling corresponding **MPI_Waitall**.
- **MPI_Isend** and **MPI_Irecv** can be synchronized simultaneously with a single **MPI_Waitall** if it is consitent.
  - Same **request** should be used in **MPI_Isend** and **MPI_Irecv**.
- Its operation is similar to that of **MPI_Barrier** but, **MPI_Waitall** can not be replaced by **MPI_Barrier.**
  - Possible troubles using **MPI_Barrier** instead of **MPI_Waitall**: Contents of **request** and **status** are not updated properly, very slow operations etc.

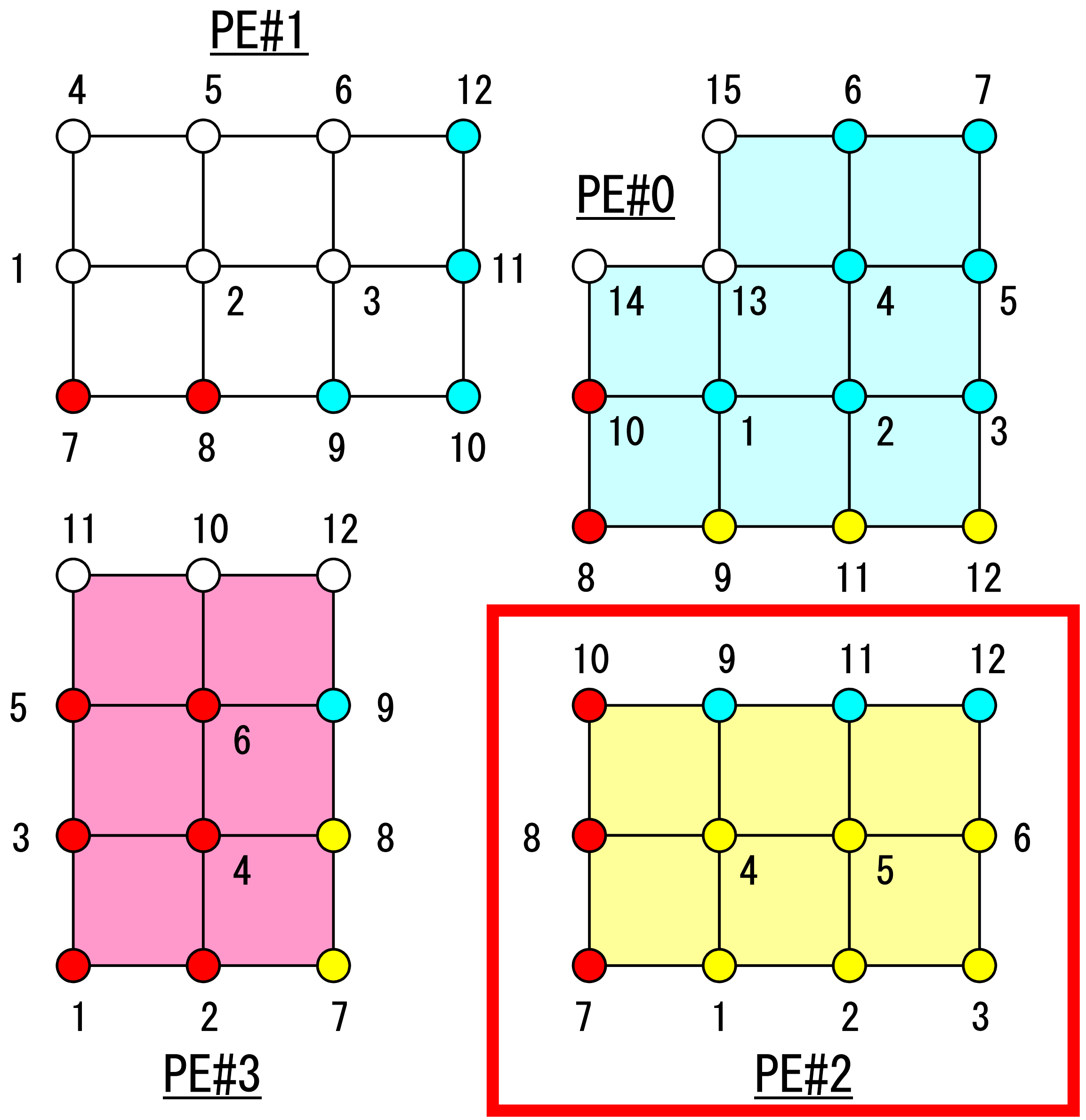
- **MPI_Waitall (count,request,status)**

– **count**	int	I	number of processes to be synchronized
– **request**	MPI_Request	I/O	comm. request used in MPI_Waitall (array size: count)
– **status**	MPI_Status	O	array of status objects MPI_STATUS_SIZE: defined in 'mpif.h', 'mpi.h'

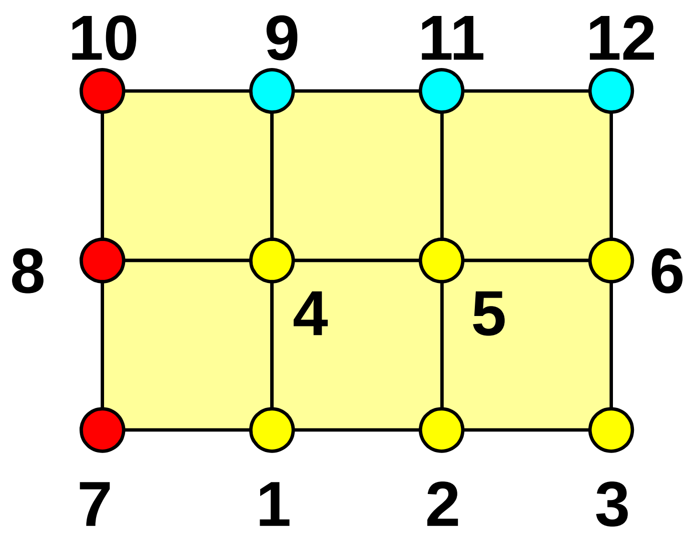
Node-based Partitioning

internal nodes - elements - external nodes





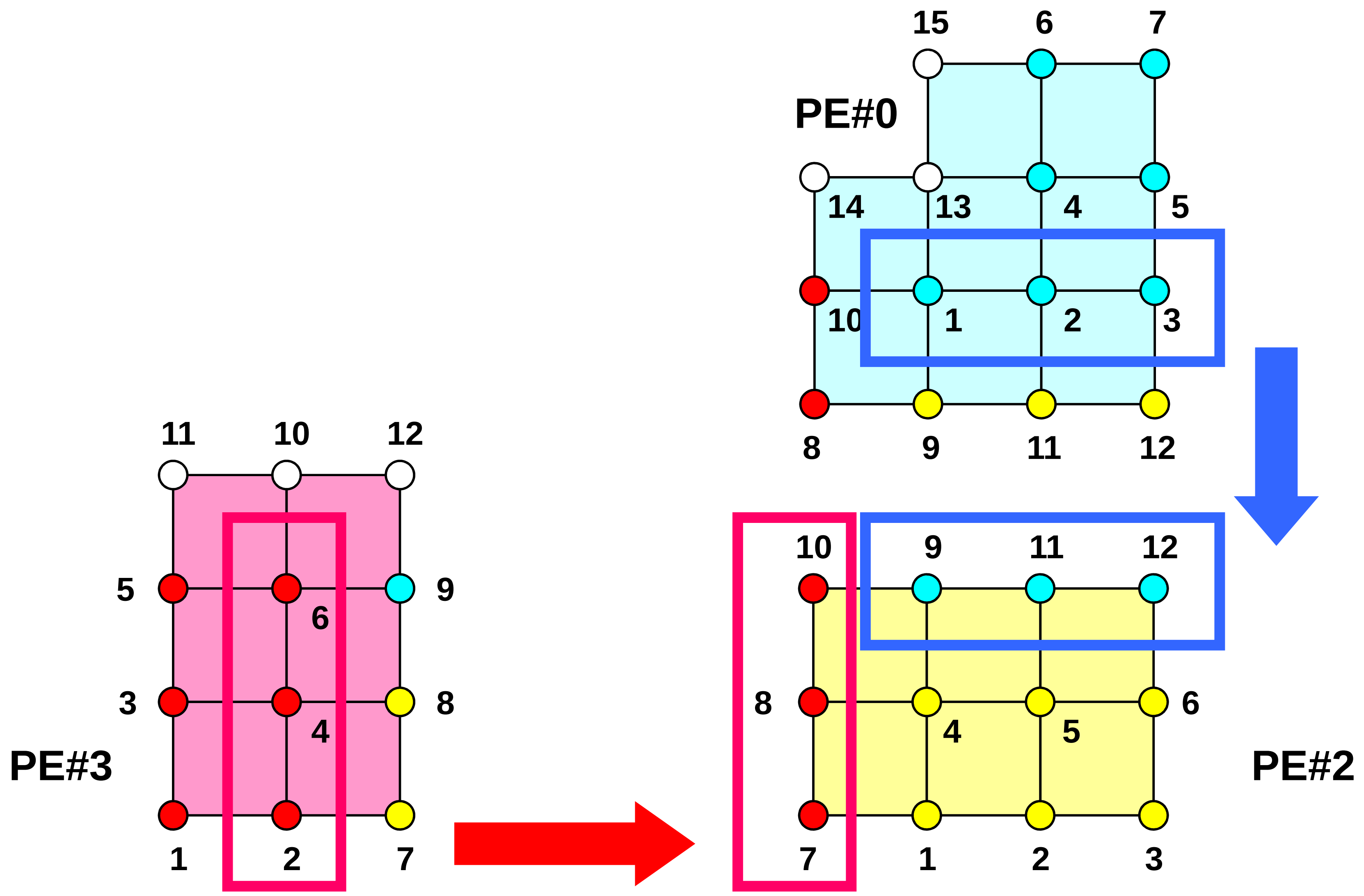
Description of Distributed Local Data



- Internal/External Points
  - Numbering: Starting from internal pts, then external pts after that
- Neighbors
  - Shares overlapped meshes
  - Number and ID of neighbors
- External Points
  - From where, how many, and which external points are received/imported ?
- Boundary Points
  - To where, how many and which boundary points are sent/exported ?

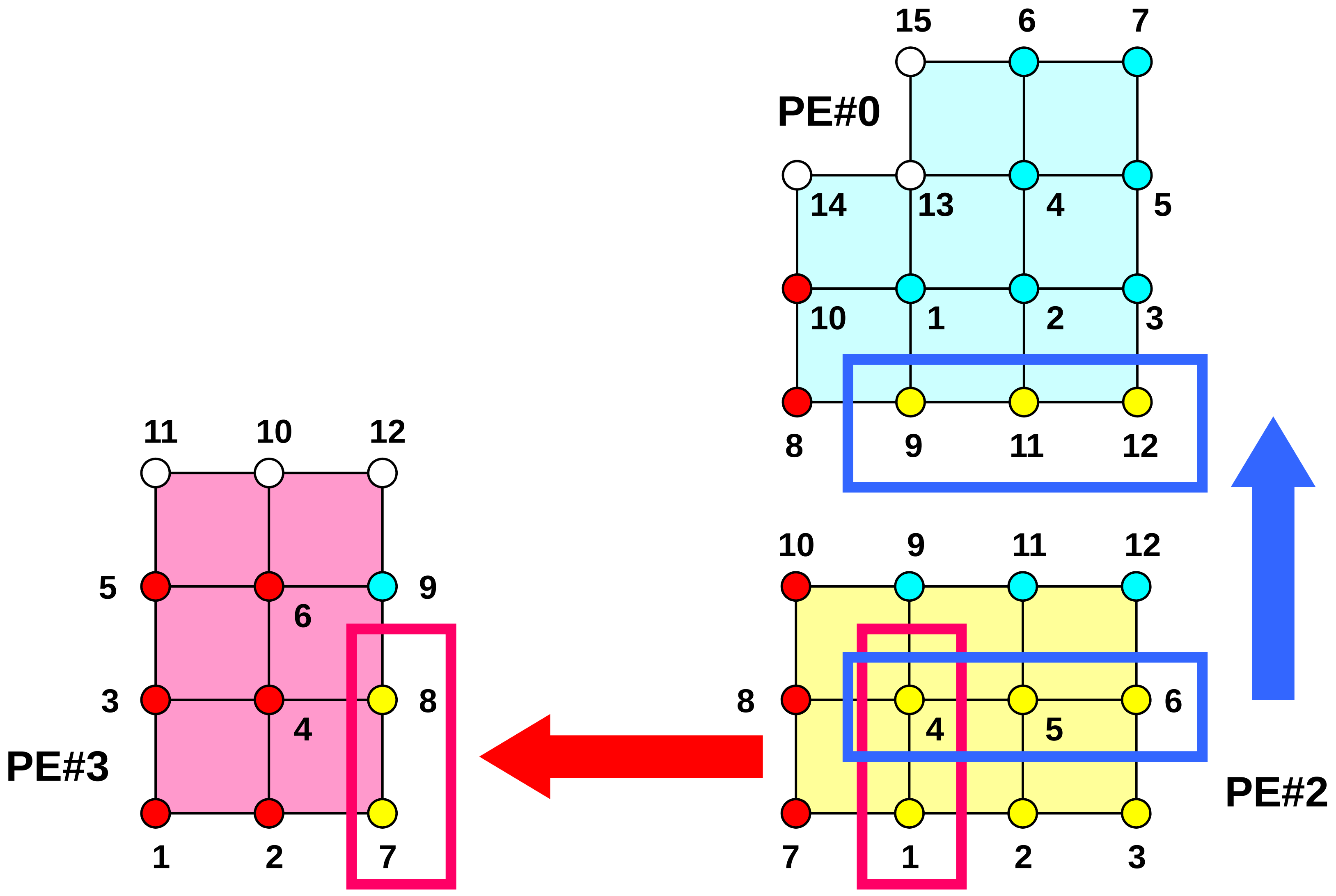
External Nodes（外点）: RECEIVE

PE#2 : receive information for “external nodes”



Boundary Nodes（境界点）: SEND

PE#2 : send information on “boundary nodes”



Distributed Local Data Structure for Parallel Computation

- Distributed local data structure for domain-to-doain communications has been introduced, which is appropriate for such applications with sparse coefficient matrices (e.g. FDM, FEM, FVM etc.).
  - SPMD
  - Local Numbering: Internal pts to External pts
  - Generalized communication table

- Everything is easy, if proper data structure is defined:
  - Values at boundary pts are copied into sending buffers
  - Send/Recv
  - Values at external pts are updated through receiving buffers