Preconditioning Methods for Iterative Solvers

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Preconditioning for Iterative Solvers

- Convergence rate of iterative solvers strongly depends on the spectral properties (eigenvalue distribution) of the coefficient matrix **A**.
  - Eigenvalue distribution is small, eigenvalues are close to 1
  - In "ill-conditioned" problems, "condition number" (ratio of max/min eigenvalue if **A** is symmetric) is large（条件数）.
- A preconditioner **M** (whose properties are similar to those of **A**) transforms the linear system into one with more favorable spectral properties（前処理）
  - **M** transforms $\mathbf{Ax=b}$ into $\mathbf{A'x=b'}$ where $\mathbf{A'=M^{-1}A}$, $\mathbf{b'=M^{-1}b}$
  - If $\mathbf{M \sim A}$, $\mathbf{M^{-1}A}$ is close to identity matrix.
  - If $\mathbf{M^{-1}=A^{-1}}$, this is the best preconditioner (Gaussian Elim.)
  - Generally, $\mathbf{A'x'=b'}$ where $\mathbf{A'=M_L^{-1}AM_R^{-1}}$, $\mathbf{b'=M_L^{-1}b}$, $\mathbf{x'=M_Rx}$
  - $\mathbf{M_L/M_R}$: Left/Right Preconditioning（左／右前処理）

Preconditioned CG Solver (PCG)

```
Compute r(0)= b-[A]x(0)
for i= 1, 2, …
    solve [M]z(i-1)= r(i-1)
    ρi-1= r(i-1) z(i-1)
    if i=1
      p(1)= z(0)
     else
      βi-1= ρi-1/ρi-2
      p(i)= z(i-1) + βi-1 p(i-1)
    endif
    q(i)= [A]p(i)
    αi  = ρi-1/p(i)q(i)
    x(i)= x(i-1) + αip(i)
    r(i)= r(i-1) - αiq(i)
    check convergence |r|
end
```

$[M]= [M_1][M_2]$

$[A']x'=b'$
$[A']=[M_1]^{-1}[A][M_2]^{-1}$
$x'=[M_2]x,\ b'=[M_1]^{-1}b$

$p'=>[M_2]p,\ r'=>[M_1]^{-1}r$

$p'^{(i)}= r'^{(i-1)} + \beta'_{i-1}\ p'^{(i-1)}$

$[M_2]p^{(i)}= [M_1]^{-1}r^{(i-1)} + \beta'_{i-1}\ [M_2]p^{(i-1)}$

$p^{(i)}= [M_2]^{-1}[M_1]^{-1}r^{(i-1)} + \beta'_{i-1}\ p^{(i-1)}$
$p^{(i)}= [M]^{-1}r^{(i-1)} + \beta'_{i-1}\ p^{(i-1)}$

$$\beta'_{i-1}= ([M]^{-1}r^{(i-1)},r^{(i-1)})/([M]^{-1}r^{(i-2)},r^{(i-2)})$$
$$\alpha'_{i-1}= ([M]^{-1}r^{(i-1)},r^{(i-1)})/(\ p^{(i-1)},[A]p^{(i-1)})$$

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    q(i)= [A]p(i)
    αi  = ρi-1/p(i)q(i)
    x(i)= x(i-1) + αip(i)
    r(i)= r(i-1) - αiq(i)
    check convergence |r|
end
```

Solving the following equation:

$$\{z\} = [M]^{-1}\{r\}$$

"Approximate Inverse Matrix"
(近似逆行列)

$$[M]^{-1} \approx [A]^{-1}, \quad [M] \approx [A]$$

Ultimate Preconditioning:
Inverse Matrix

$$[M]^{-1} = [A]^{-1}, \quad [M] = [A]$$

Diagonal Scaling: Simple but weak

$$[M]^{-1} = [D]^{-1}, \quad [M] = [D]$$

Diagonal Scaling, Point-Jacobi

$$
[M]=\begin{bmatrix}
D_1 & 0 & \dots & 0 & 0 \\
0 & D_2 & & 0 & 0 \\
\dots & & \dots & & \dots \\
0 & 0 & & D_{N-1} & 0 \\
0 & 0 & \dots & 0 & D_N
\end{bmatrix}
$$

- **solve [M]z$^{(i-1)}$= r$^{(i-1)}$** is very easy.
- Provides fast convergence for simple problems.
- 1d.f, 1d.c

ILU(0), IC(0)

- Widely used Preconditioners for Sparse Matrices
  - Incomplete LU Factorization（不完全LU分解）
  - Incomplete Cholesky Factorization (for Symmetric Matrices)（不完全コレスキー分解）

- Incomplete Direct Method
  - Even if original matrix is sparse, inverse matrix is not necessarily sparse.
  - fill-in
  - ILU(0)/IC(0) without fill-in have same non-zero pattern with the original (sparse) matrices

LU Factorization/Decomposition: Complete LU Factorization LU分解・完全LU分解

- A kind of direct method for solving linear eqn's
  - compute "inverse matrix" directly
  - Information of "inverse matrix" can be saved, therefore it's efficient for multiple RHS cases
  - "Fill-in" may occur during factorization/decomposition
    - entries which change from an initial zero to a non-zero value during the execution of factorization/decomposition
- LU factorization

Incomplete LU Factorization

- ILU factorization
  - Incomplete LU factorization
- Preconditioning method using “incomplete” inverse matrices, where generation of “fill-in” is controlled
  - Approximate/Incomplete Inverse Matrix, Weak Direct method
  - ILU(0): NO fill-in is allowed

Solving Linear Equations by LU Factorization

LU factorization of matrix A (n×n):

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ l_{21} & 1 & 0 & \cdots & 0 \\ l_{31} & l_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2n} \\ 0 & 0 & u_{33} & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & u_{nn} \end{pmatrix}$$

$$\mathbf{A} = \mathbf{LU}$$

L : Lower triangular part of matrix A

U : Upper triangular part of matrix A

Matrix Form of Linear Equation

General Form of Linear Equation with “n” unknowns

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

$$\Updownarrow$$

Matdix Form

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \Leftrightarrow \mathbf{Ax} = \mathbf{b}$$

$$\mathbf{A} \qquad \mathbf{x} \qquad \mathbf{b}$$

Solving Ax=b by LU Factorization

1 $\mathbf{A} = \mathbf{LU}$ LU factorization of A

2 $\mathbf{Ly} = \mathbf{b}$ Compute {y} (easy)

3 $\mathbf{Ux} = \mathbf{y}$ Compute {x} (easy)

This {x} satisfies $\mathbf{Ax} = \mathbf{b}$

$$\because \mathbf{Ax} = \mathbf{LUx} = \mathbf{Ly} = \mathbf{b}$$

Forward Substitution：前進代入
Solving Ly=b

$$\mathbf{Ly} = \mathbf{b} \quad \Leftrightarrow \quad \begin{pmatrix} 1 & 0 & \cdots & 0 \\ l_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \cdots & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$\begin{aligned} y_1 &= b_1 \\ l_{21} y_1 + y_2 &= b_2 \\ &\vdots \\ l_{n1} y_1 + l_{n2} y_2 + \cdots + y_n &= b_n \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} y_1 &= b_1 \\ y_2 &= b_2 - l_{21} y_1 \\ &\vdots \\ y_n &= b_n - l_{n1} y_1 - l_{n2} y_2 = b_n - \sum_{i=1}^{n-1} l_{ni} y_i \end{aligned}$$

row-by-row substitutio

Backward Substitution：後退代入
Solving Ux=y

$$\mathbf{Ux} = \mathbf{y} \quad \Longleftrightarrow \quad \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

$$\begin{aligned} & u_{nn} x_n = y_n \\ & u_{n-1,n-1} x_{n-1} + u_{n-1,n} x_n = y_{n-1} \\ & \vdots \\ & u_{11} x_1 + u_{12} x_2 + \cdots + u_{1n} x_n = y_1 \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} & x_n = y_n / u_{nn} \\ & x_{n-1} = (y_{n-1} - u_{n-1,n} x_n) / u_{n-1,n-1} \\ & \vdots \\ & x_1 = \left(y_1 - \sum_{i=2}^{n} u_{1j} x_j \right) / u_{11} \end{aligned}$$

row−by−row substitutio

Computation of LU Factorization

$$
\begin{pmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\
a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}
\end{pmatrix}
=
\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
l_{21} & 1 & 0 & \cdots & 0 \\
l_{31} & l_{32} & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
l_{n1} & l_{n2} & l_{n3} & \cdots & 1
\end{pmatrix}
\begin{pmatrix}
u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\
0 & u_{22} & u_{23} & \cdots & u_{2n} \\
0 & 0 & u_{33} & \cdots & u_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & u_{nn}
\end{pmatrix}
$$

(1) ➡ $a_{11} = u_{11}, a_{12} = u_{12}, \cdots, a_{1n} = u_{1n} \Rightarrow u_{11}, u_{12}, \cdots, u_{1n}$

(2) ➡ $a_{21} = l_{21}u_{11}, a_{31} = l_{31}u_{11}, \cdots, a_{n1} = l_{n1}u_{11} \Rightarrow l_{21}, l_{31}, \cdots, l_{n1}$

(3) ➡ $a_{22} = l_{21}u_{12} + u_{22}, \cdots, a_{2n} = l_{21}u_{1n} + u_{2n} \Rightarrow u_{22}, u_{23}, \cdots, u_{2n}$

(4) ➡ $a_{32} = l_{31}u_{12} + l_{32}u_{22}, \cdots \Rightarrow l_{32}, l_{42}, \cdots, l_{n2}$

Example

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 6 & 7 & 10 \\ 2 & 2 & 8 & 7 \\ 0 & -4 & 7 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{pmatrix}$$

1st row ➡ $1 = u_{11}, 2 = u_{12}, 3 = u_{13}, 4 = u_{14}$

1st col. ➡ $2 = l_{21}u_{11} \Rightarrow l_{21} = 2/u_{11} = 2, \quad 2 = l_{31}u_{11} \Rightarrow l_{31} = 2/u_{11} = 2$

$$0 = l_{41}u_{11} \Rightarrow l_{41} = 0/u_{11} = 0$$

2nd row ➡ $6 = l_{21}u_{12} + u_{22} \Rightarrow u_{22} = 2, \quad 7 = l_{21}u_{13} + u_{23} \Rightarrow u_{23} = 1$

$$10 = l_{21}u_{14} + u_{24} \Rightarrow u_{24} = 2$$

2nd col. ➡ $2 = l_{31}u_{12} + l_{32}u_{22} \Rightarrow l_{32} = -1, \quad -4 = l_{41}u_{12} + l_{42}u_{22} \Rightarrow l_{42} = -2$

Example (cont.)

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 6 & 7 & 10 \\ 2 & 2 & 8 & 7 \\ 0 & -4 & 7 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{pmatrix}$$

3rd row ➡

$$8 = l_{31}u_{13} + l_{32}u_{23} + u_{33} \Rightarrow u_{33} = 3,$$
$$7 = l_{31}u_{14} + l_{32}u_{24} + u_{34} \Rightarrow u_{34} = 1$$

3rd col ➡

$$7 = l_{41}u_{13} + l_{42}u_{23} + l_{43}u_{33} \Rightarrow l_{43} = 3$$

4th row/col ➡

$$1 = l_{41}u_{14} + l_{42}u_{24} + l_{43}u_{34} + u_{44} \Rightarrow u_{44} = 2$$

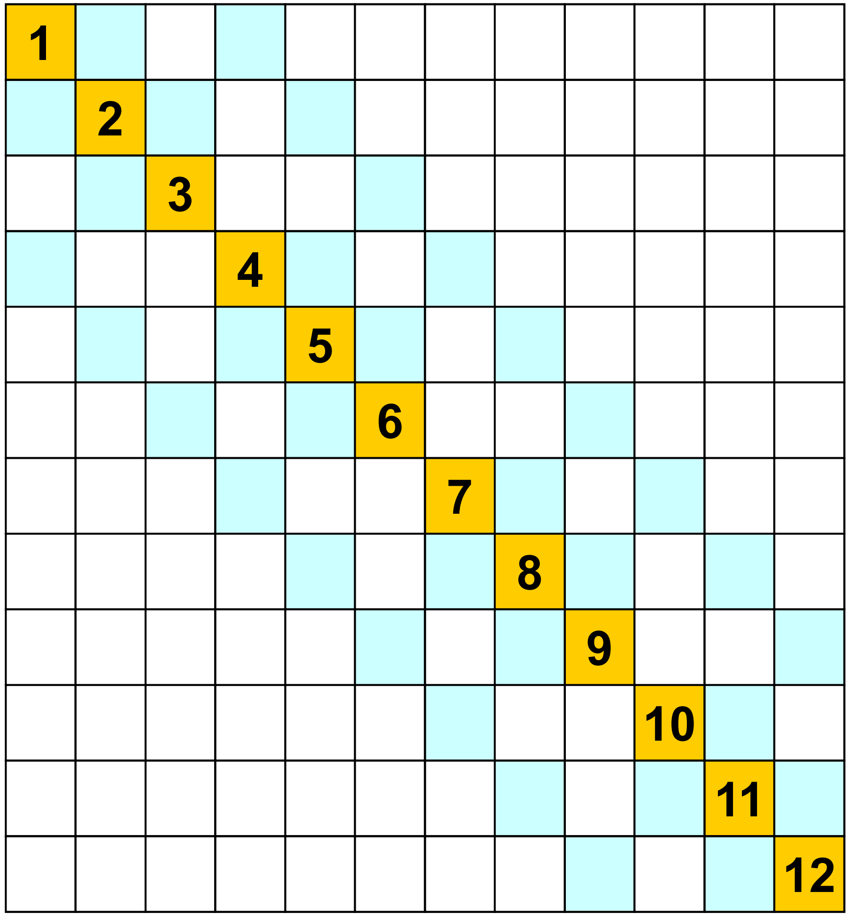
Solving according to 1st row–column, 2nd row–column, 3rd row–column ···

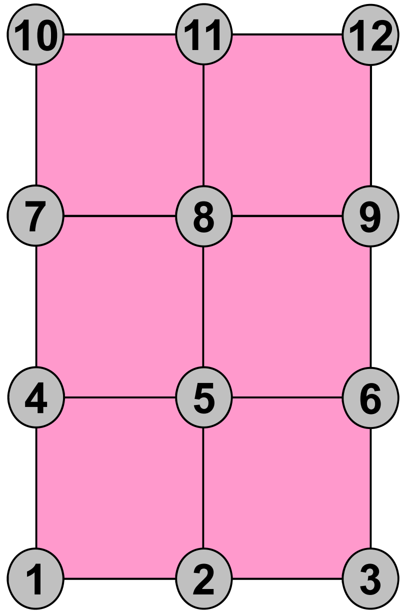
Example (cont.)

Finally:

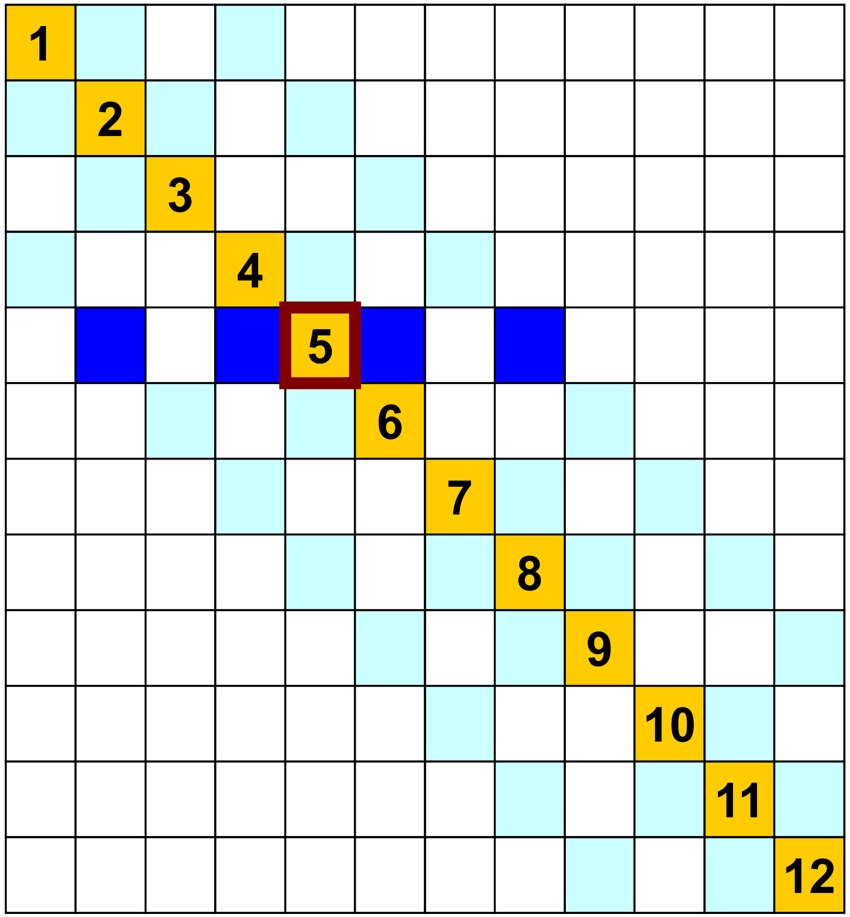
$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 6 & 7 & 10 \\ 2 & 2 & 8 & 7 \\ 0 & -4 & 7 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 \\ 0 & -2 & 3 & 1 \end{pmatrix}}_{\mathbf{L}} \underbrace{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}}_{\mathbf{U}}$$

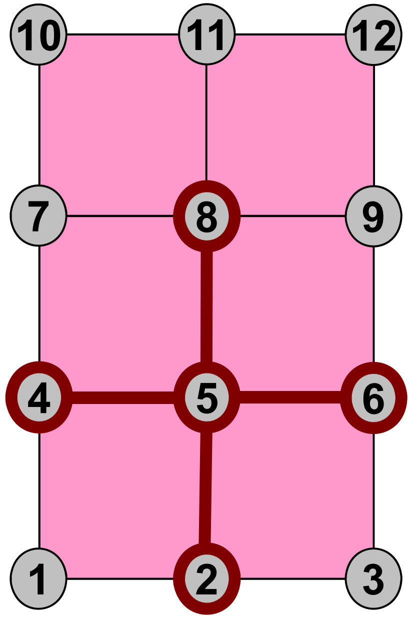
Example: 5-Point Stencil (FDM)
五点差分





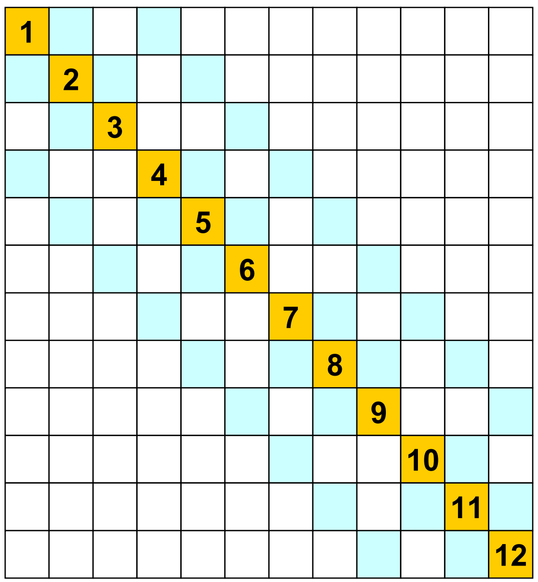
Example: 5-Point Stencil (FDM)

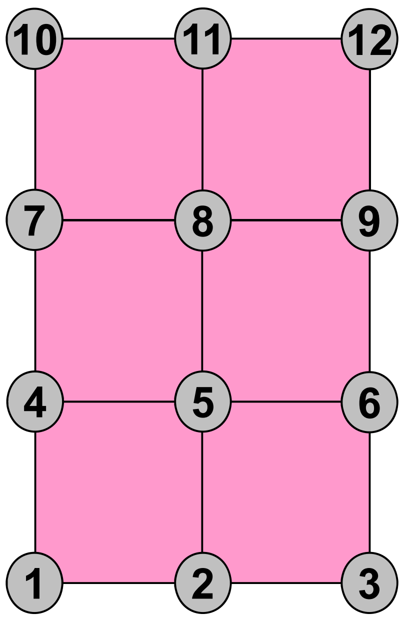




Coef. Matrix: Diag. Component=6.00

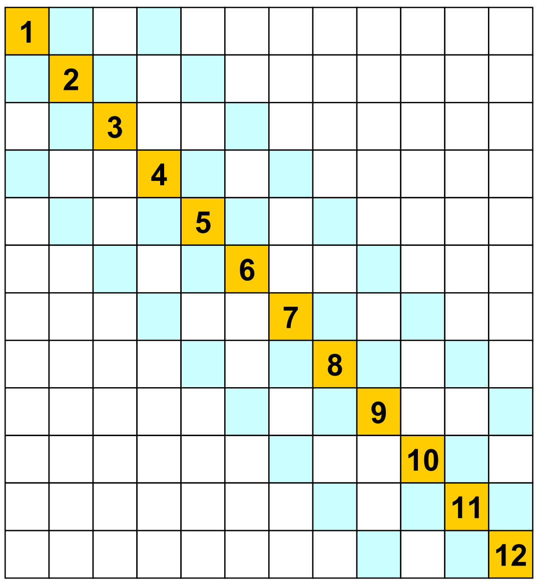
$$
\begin{pmatrix}
6.00 & -1.00 & 0.00 & -1.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
-1.00 & 6.00 & -1.00 & 0.00 & -1.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.00 & -1.00 & 6.00 & 0.00 & 0.00 & -1.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
-1.00 & 0.00 & 0.00 & 6.00 & -1.00 & 0.00 & -1.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.00 & -1.00 & 0.00 & -1.00 & 6.00 & -1.00 & 0.00 & -1.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.00 & -1.00 & 0.00 & -1.00 & 6.00 & 0.00 & 0.00 & -1.00 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.00 & 0.00 & -1.00 & 0.00 & 0.00 & 6.00 & -1.00 & 0.00 & -1.00 & 0.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & -1.00 & 0.00 & -1.00 & 6.00 & -1.00 & 0.00 & -1.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & -1.00 & 0.00 & -1.00 & 6.00 & 0.00 & 0.00 & -1.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & -1.00 & 0.00 & 0.00 & 6.00 & -1.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & -1.00 & 0.00 & -1.00 & 6.00 & -1.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & -1.00 & 0.00 & -1.00 & 6.00
\end{pmatrix}
\begin{pmatrix} \\ \\ \\ \\ \\ X \\ \\ \\ \\ \\ \\ \end{pmatrix}
=
\begin{pmatrix}
0.00 \\ 3.00 \\ 10.00 \\ 11.00 \\ 10.00 \\ 19.00 \\ 20.00 \\ 16.00 \\ 28.00 \\ 42.00 \\ 36.00 \\ 52.00
\end{pmatrix}
$$

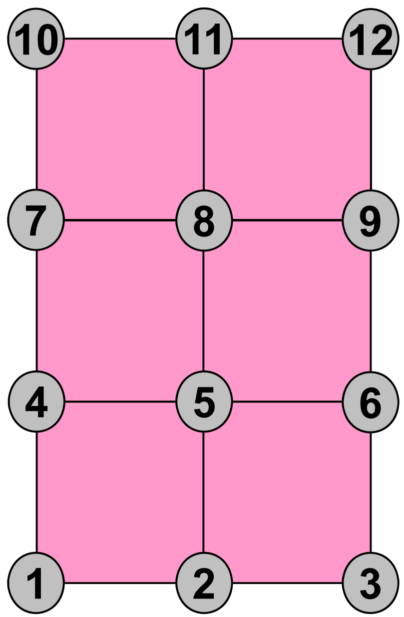




Solution

$$
\begin{pmatrix}
6.00 & -1.00 & 0.00 & -1.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
-1.00 & 6.00 & -1.00 & 0.00 & -1.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.00 & -1.00 & 6.00 & 0.00 & 0.00 & -1.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
-1.00 & 0.00 & 0.00 & 6.00 & -1.00 & 0.00 & -1.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.00 & -1.00 & 0.00 & -1.00 & 6.00 & -1.00 & 0.00 & -1.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.00 & -1.00 & 0.00 & -1.00 & 6.00 & 0.00 & 0.00 & -1.00 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.00 & 0.00 & -1.00 & 0.00 & 0.00 & 6.00 & -1.00 & 0.00 & -1.00 & 0.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & -1.00 & 0.00 & -1.00 & 6.00 & -1.00 & 0.00 & -1.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & -1.00 & 0.00 & -1.00 & 6.00 & 0.00 & 0.00 & -1.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & -1.00 & 0.00 & 0.00 & 6.00 & -1.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & -1.00 & 0.00 & -1.00 & 6.00 & -1.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & -1.00 & 0.00 & -1.00 & 6.00
\end{pmatrix}
\begin{pmatrix}
1.00 \\ 2.00 \\ 3.00 \\ 4.00 \\ 5.00 \\ 6.00 \\ 7.00 \\ 8.00 \\ 9.00 \\ 10.00 \\ 11.00 \\ 12.00
\end{pmatrix}
=
\begin{pmatrix}
0.00 \\ 3.00 \\ 10.00 \\ 11.00 \\ 10.00 \\ 19.00 \\ 20.00 \\ 16.00 \\ 28.00 \\ 42.00 \\ 36.00 \\ 52.00
\end{pmatrix}
$$





Complete LU Factorization

type "./lu1"

Original Matrix

6.00	−1.00	0.00	−1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
−1.00	6.00	−1.00	0.00	−1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.00	−1.00	6.00	0.00	0.00	−1.00	0.00	0.00	0.00	0.00	0.00	0.00
−1.00	0.00	0.00	6.00	−1.00	0.00	−1.00	0.00	0.00	0.00	0.00	0.00
0.00	−1.00	0.00	−1.00	6.00	−1.00	0.00	−1.00	0.00	0.00	0.00	0.00
0.00	0.00	−1.00	0.00	−1.00	6.00	0.00	0.00	−1.00	0.00	0.00	0.00
0.00	0.00	0.00	−1.00	0.00	0.00	6.00	−1.00	0.00	−1.00	0.00	0.00
0.00	0.00	0.00	0.00	−1.00	0.00	−1.00	6.00	−1.00	0.00	−1.00	0.00
0.00	0.00	0.00	0.00	0.00	−1.00	0.00	−1.00	6.00	0.00	0.00	−1.00
0.00	0.00	0.00	0.00	0.00	0.00	−1.00	0.00	0.00	6.00	−1.00	0.00
0.00	0.00	0.00	0.00	0.00	0.00	0.00	−1.00	0.00	−1.00	6.00	−1.00
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	−1.00	0.00	−1.00	6.00

LU Factorization

Both of [L] and [U] are shown
Diag. of [L] are "1" (not shown)

fill-in occurs: some of zero components became non-zero.

6.00	−1.00	0.00	−1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
−0.17	5.83	−1.00	−0.17	−1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.00	−0.17	5.83	−0.03	−0.17	−1.00	0.00	0.00	0.00	0.00	0.00	0.00
−0.17	−0.03	0.00	5.83	−1.03	0.00	−1.00	0.00	0.00	0.00	0.00	0.00
0.00	−0.17	−0.03	−0.18	5.64	−1.03	−0.18	−1.00	0.00	0.00	0.00	0.00
0.00	0.00	−0.17	0.00	−0.18	5.64	−0.03	−0.18	−1.00	0.00	0.00	0.00
0.00	0.00	0.00	−0.17	−0.03	−0.01	5.82	−1.03	−0.01	−1.00	0.00	0.00
0.00	0.00	0.00	0.00	−0.18	−0.03	−0.18	5.63	−1.03	−0.18	−1.00	0.00
0.00	0.00	0.00	0.00	0.00	−0.18	0.00	−0.18	5.63	−0.03	−0.18	−1.00
0.00	0.00	0.00	0.00	0.00	0.00	−0.17	−0.03	−0.01	5.82	−1.03	−0.01
0.00	0.00	0.00	0.00	0.00	0.00	0.00	−0.18	−0.03	−0.18	5.63	−1.03
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	−0.18	0.00	−0.18	5.63

Incomp. LU fact. with no fill-in's

type "./lu2"

Incomplete LU Factorization without fill-in's

Both of [L] and [U] are shown

Diag. of [L] are "1" (not shown)

6.00	−1.00	0.00	−1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
−0.17	5.83	−1.00	0.00	−1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.00	−0.17	5.83	0.00	0.00	−1.00	0.00	0.00	0.00	0.00	0.00	0.00
−0.17	0.00	0.00	5.83	−1.00	0.00	−1.00	0.00	0.00	0.00	0.00	0.00
0.00	−0.17	0.00	−0.17	5.66	−1.00	0.00	−1.00	0.00	0.00	0.00	0.00
0.00	0.00	−0.17	0.00	−0.18	5.65	0.00	0.00	−1.00	0.00	0.00	0.00
0.00	0.00	0.00	−0.17	0.00	0.00	5.83	−1.00	0.00	−1.00	0.00	0.00
0.00	0.00	0.00	0.00	−0.18	0.00	−0.17	5.65	−1.00	0.00	−1.00	0.00
0.00	0.00	0.00	0.00	0.00	−0.18	0.00	−0.18	5.65	0.00	0.00	−1.00
0.00	0.00	0.00	0.00	0.00	0.00	−0.17	0.00	0.00	5.83	−1.00	0.00
0.00	0.00	0.00	0.00	0.00	0.00	0.00	−0.18	0.00	−0.17	5.65	−1.00
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	−0.18	0.00	−0.18	5.65

LU Factorization

Both of [L] and [U] are shown

Diag. of [L] are "1" (not shown)

fill-in occurs: some of zero components became non-zero.

6.00	−1.00	0.00	−1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
−0.17	5.83	−1.00	−0.17	−1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.00	−0.17	5.83	−0.03	−0.17	−1.00	0.00	0.00	0.00	0.00	0.00	0.00
−0.17	−0.03	0.00	5.83	−1.03	0.00	−1.00	0.00	0.00	0.00	0.00	0.00
0.00	−0.17	−0.03	−0.18	5.64	−1.03	−0.18	−1.00	0.00	0.00	0.00	0.00
0.00	0.00	−0.17	0.00	−0.18	5.64	−0.03	−0.18	−1.00	0.00	0.00	0.00
0.00	0.00	0.00	−0.17	−0.03	−0.01	5.82	−1.03	−0.01	−1.00	0.00	0.00
0.00	0.00	0.00	0.00	−0.18	−0.03	−0.18	5.63	−1.03	−0.18	−1.00	0.00
0.00	0.00	0.00	0.00	0.00	−0.18	0.00	−0.18	5.63	−0.03	−0.18	−1.00
0.00	0.00	0.00	0.00	0.00	0.00	−0.17	−0.03	−0.01	5.82	−1.03	−0.01
0.00	0.00	0.00	0.00	0.00	0.00	0.00	−0.18	−0.03	−0.18	5.63	−1.03
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	−0.18	0.00	−0.18	5.63

Slightly "Inaccurate" Solution

Incomplete LU

6.00	-1.00	0.00	-1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
-0.17	5.83	-1.00	0.00	-1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.00	-0.17	5.83	0.00	0.00	-1.00	0.00	0.00	0.00	0.00	0.00	0.00
-0.17	0.00	0.00	5.83	-1.00	0.00	-1.00	0.00	0.00	0.00	0.00	0.00
0.00	-0.17	0.00	-0.17	5.66	-1.00	0.00	-1.00	0.00	0.00	0.00	0.00
0.00	0.00	-0.17	0.00	-0.18	5.65	0.00	0.00	-1.00	0.00	0.00	0.00
0.00	0.00	0.00	-0.17	0.00	0.00	5.83	-1.00	0.00	-1.00	0.00	0.00
0.00	0.00	0.00	0.00	-0.18	0.00	-0.17	5.65	-1.00	0.00	-1.00	0.00
0.00	0.00	0.00	0.00	0.00	-0.18	0.00	-0.18	5.65	0.00	0.00	-1.00
0.00	0.00	0.00	0.00	0.00	0.00	-0.17	0.00	0.00	5.83	-1.00	0.00
0.00	0.00	0.00	0.00	0.00	0.00	0.00	-0.18	0.00	-0.17	5.65	-1.00
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	-0.18	0.00	-0.18	5.65

0.92
1.75
2.76
3.79
4.46
5.57
6.66
7.25
8.46
9.66
10.54
11.83

Complete LU

6.00	-1.00	0.00	-1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
-0.17	5.83	-1.00	-0.17	-1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.00	-0.17	5.83	-0.03	-0.17	-1.00	0.00	0.00	0.00	0.00	0.00	0.00
-0.17	-0.03	0.00	5.83	-1.03	0.00	-1.00	0.00	0.00	0.00	0.00	0.00
0.00	-0.17	-0.03	-0.18	5.64	-1.03	-0.18	-1.00	0.00	0.00	0.00	0.00
0.00	0.00	-0.17	0.00	-0.18	5.64	-0.03	-0.18	-1.00	0.00	0.00	0.00
0.00	0.00	0.00	-0.17	-0.03	-0.01	5.82	-1.03	-0.01	-1.00	0.00	0.00
0.00	0.00	0.00	0.00	-0.18	-0.03	-0.18	5.63	-1.03	-0.18	-1.00	0.00
0.00	0.00	0.00	0.00	0.00	-0.18	0.00	-0.18	5.63	-0.03	-0.18	-1.00
0.00	0.00	0.00	0.00	0.00	0.00	-0.17	-0.03	-0.01	5.82	-1.03	-0.01
0.00	0.00	0.00	0.00	0.00	0.00	0.00	-0.18	-0.03	-0.18	5.63	-1.03
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	-0.18	0.00	-0.18	5.63

1.00
2.00
3.00
4.00
5.00
6.00
7.00
8.00
9.00
10.00
11.00
12.00

ILU(0), IC(0)

- "Incomplete" factorization without fill-in's
  - Reduced memory, computation
- Solving equations by ILU(0)/IC(0) factorization provides slightly "inaccurate" solution, although it's not far from exact one.
  - "Accurateness" depends on problems (feature of equations).

Full LU and ILU(0)/IC(0)

Full LU

```
do i= 2, n
  do k= 1,  i-1
    a_ik := a_ik/a_kk
    do j= k+1, n
      a_ij := a_ij - a_ik*a_kj
    enddo
  enddo
enddo
```

ILU(0) : keep non-zero pattern of the original coefficient matrix

```
do i= 2, n
  do k= 1,  i-1
    if ((i,k)∈ NonZero(A)) then
      a_ik := a_ik/a_kk
    endif
    do j= k+1, n
      if ((i,j)∈ NonZero(A)) then
        a_ij := a_ij - a_ik*a_kj
      endif
    enddo
  enddo
enddo
```

Deep Fill-in: ILU(p)/IC(p)

p: level of fill-in. If “p” increases, ILU(p)/IC(p) become closer to complete ILU/IC and provide more robust preconditioners, but become more expensive: trade-off

```
LEV_ij=0 if ((i,j)∈ NonZero(A)) otherwise LEV_ij= p+1

do i= 2, n
  do k= 1,  i-1
    if (LEV_ik≦p) then
      a_ik := a_ik/a_kk
    endif
    do j= k+1, n
      if (LEV_ij = min(LEV_ij, 1+LEV_ik+ LEV_kj)≦p) then
        a_ij := a_ij - a_ik*a_kj
      endif
    enddo
  enddo
enddo
```

LU Gauss-Seidel (LU-GS)
LU Symmetric GS (LU-SGS)
in this class

- ILU(0)

```
do i= 2, n
    do k= 1,  i-1
      if ((i,k)∈ NonZero(A)) then
        a_ik := a_ik/a_kk
      endif
      do j= k+1, n
        if ((i,j)∈ NonZero(A)) then
          a_ij := a_ij - a_ik*a_kj
        endif
      enddo
    enddo
  enddo
```

LU Gauss-Seidel (LU-GS)
LU Symmetric GS (LU-SGS)
in this class

- More Simplified Version of ILU(0)

```
do i= 2, n
   do k= 1,  i-1
     if ((i,k)∈ NonZero(A)) then
       a_ik := a_ik/a_kk
     endif
     do j= k+1, n
       if ((i,j)∈ NonZero(A)) then
         a_ij := a_ij - a_ik*a_kj
       endif
     enddo
   enddo
 enddo
```

Only do this

LU Gauss-Seidel (LU-GS)
LU Symmetric GS (LU-SGS)
in this class

- More Simplified Version of ILU(0)

$$
\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
l_{21} & 1 & 0 & \cdots & 0 \\
l_{31} & l_{32} & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
l_{n1} & l_{n2} & l_{n3} & \cdots & 1
\end{pmatrix}
=
\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
a_{21}/a_{22} & 1 & 0 & \cdots & 0 \\
a_{31}/a_{33} & a_{32}/a_{33} & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{n1}/a_{nn} & a_{n2}/a_{nn} & a_{n3}/a_{nn} & \cdots & 1
\end{pmatrix}
$$

$$
\begin{pmatrix}
u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\
0 & u_{22} & u_{23} & \cdots & u_{2n} \\
0 & 0 & u_{33} & \cdots & u_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & u_{nn}
\end{pmatrix}
=
\begin{pmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
0 & a_{22} & a_{23} & \cdots & a_{2n} \\
0 & 0 & a_{33} & \cdots & a_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & a_{nn}
\end{pmatrix}
$$

```
do i= 2, n
   do k= 1,  i-1
     if ((i,k)∈ NonZero(A)) then
       a_ik := a_ik/a_kk
     endif
     do j= k+1, n
       if ((i,j)∈ NonZero(A)) then
         a_ij := a_ij - a_ik*a_kj
       endif
     enddo
   enddo
enddo
```

ILU, LU-GS

type "./lu3"

Incomplete LU Factorization without Fill-in's

6.00	-1.00	0.00	-1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
-0.17	5.83	-1.00	0.00	-1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.00	-0.17	5.83	0.00	0.00	-1.00	0.00	0.00	0.00	0.00	0.00	0.00
-0.17	0.00	0.00	5.83	-1.00	0.00	-1.00	0.00	0.00	0.00	0.00	0.00
0.00	-0.17	0.00	-0.17	5.66	-1.00	0.00	-1.00	0.00	0.00	0.00	0.00
0.00	0.00	-0.17	0.00	-0.18	5.65	0.00	0.00	-1.00	0.00	0.00	0.00
0.00	0.00	0.00	-0.17	0.00	0.00	5.83	-1.00	0.00	-1.00	0.00	0.00
0.00	0.00	0.00	0.00	-0.18	0.00	-0.17	5.65	-1.00	0.00	-1.00	0.00
0.00	0.00	0.00	0.00	0.00	-0.18	0.00	-0.18	5.65	0.00	0.00	-1.00
0.00	0.00	0.00	0.00	0.00	0.00	-0.17	0.00	0.00	5.83	-1.00	0.00
0.00	0.00	0.00	0.00	0.00	0.00	0.00	-0.18	0.00	-0.17	5.65	-1.00
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	-0.18	0.00	-0.18	5.65

LU-GS without Fill-in's

6.00	-1.00	0.00	-1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
-0.17	6.00	-1.00	0.00	-1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.00	-0.17	6.00	0.00	0.00	-1.00	0.00	0.00	0.00	0.00	0.00	0.00
-0.17	0.00	0.00	6.00	-1.00	0.00	-1.00	0.00	0.00	0.00	0.00	0.00
0.00	-0.17	0.00	-0.17	6.00	-1.00	0.00	-1.00	0.00	0.00	0.00	0.00
0.00	0.00	-0.17	0.00	-0.17	6.00	0.00	0.00	-1.00	0.00	0.00	0.00
0.00	0.00	0.00	-0.17	0.00	0.00	6.00	-1.00	0.00	-1.00	0.00	0.00
0.00	0.00	0.00	0.00	-0.17	0.00	-0.17	6.00	-1.00	0.00	-1.00	0.00
0.00	0.00	0.00	0.00	0.00	-0.17	0.00	-0.17	6.00	0.00	0.00	-1.00
0.00	0.00	0.00	0.00	0.00	0.00	-0.17	0.00	0.00	6.00	-1.00	0.00
0.00	0.00	0.00	0.00	0.00	0.00	0.00	-0.17	0.00	-0.17	6.00	-1.00
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	-0.17	0.00	-0.17	6.00

Solution is more “inaccurate”

ILU(0)

$$\begin{pmatrix}
6.00 & -1.00 & 0.00 & -1.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
-0.17 & 5.83 & -1.00 & 0.00 & -1.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.00 & -0.17 & 5.83 & 0.00 & 0.00 & -1.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
-0.17 & 0.00 & 0.00 & 5.83 & -1.00 & 0.00 & -1.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.00 & -0.17 & 0.00 & -0.17 & 5.66 & -1.00 & 0.00 & -1.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.00 & -0.17 & 0.00 & -0.18 & 5.65 & 0.00 & 0.00 & -1.00 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.00 & 0.00 & -0.17 & 0.00 & 0.00 & 5.83 & -1.00 & 0.00 & -1.00 & 0.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & -0.18 & 0.00 & -0.17 & 5.65 & -1.00 & 0.00 & -1.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & -0.18 & 0.00 & -0.18 & 5.65 & 0.00 & 0.00 & -1.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & -0.17 & 0.00 & 0.00 & 5.83 & -1.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & -0.18 & 0.00 & -0.17 & 5.65 & -1.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & -0.18 & 0.00 & -0.18 & 5.65
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
0.92 \\ 1.75 \\ 2.76 \\ 3.79 \\ 4.46 \\ 5.57 \\ 6.66 \\ 7.25 \\ 8.46 \\ 9.66 \\ 10.54 \\ 11.83
\end{pmatrix}$$

LU-GS

$$\begin{pmatrix}
6.00 & -1.00 & 0.00 & -1.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
-0.17 & 6.00 & -1.00 & 0.00 & -1.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.00 & -0.17 & 6.00 & 0.00 & 0.00 & -1.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
-0.17 & 0.00 & 0.00 & 6.00 & -1.00 & 0.00 & -1.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.00 & -0.17 & 0.00 & -0.17 & 6.00 & -1.00 & 0.00 & -1.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.00 & -0.17 & 0.00 & -0.17 & 6.00 & 0.00 & 0.00 & -1.00 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.00 & 0.00 & -0.17 & 0.00 & 0.00 & 6.00 & -1.00 & 0.00 & -1.00 & 0.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & -0.17 & 0.00 & -0.17 & 6.00 & -1.00 & 0.00 & -1.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & -0.17 & 0.00 & -0.17 & 6.00 & 0.00 & 0.00 & -1.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & -0.17 & 0.00 & 0.00 & 6.00 & -1.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & -0.17 & 0.00 & -0.17 & 6.00 & -1.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & -0.17 & 0.00 & -0.17 & 6.00
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
0.86 \\ 1.60 \\ 2.60 \\ 3.54 \\ 3.99 \\ 5.09 \\ 6.26 \\ 6.52 \\ 7.73 \\ 9.22 \\ 9.70 \\ 10.96
\end{pmatrix}$$

Forward/Backward Substitution in LU-GS

$$[M]\{z\} = [\tilde{L}\tilde{U}]\{z\} = \{r\}$$

$$\{z\} = [\tilde{L}\tilde{U}]^{-1}\{r\} \longrightarrow \begin{array}{l} [\tilde{L}]\{y\} = \{r\} \\ [\tilde{U}]\{z\} = \{y\} \end{array}$$

$$[\tilde{L}] = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ l_{21} & 1 & 0 & \cdots & 0 \\ l_{31} & l_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_{21}/a_{22} & 1 & 0 & \cdots & 0 \\ a_{31}/a_{33} & a_{32}/a_{33} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1}/a_{nn} & a_{n2}/a_{nn} & a_{n3}/a_{nn} & \cdots & 1 \end{pmatrix}$$

$$[\tilde{U}] = \begin{pmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2n} \\ 0 & 0 & u_{33} & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & u_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix}$$

$$
\left[\overline{L}\right]=\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ a_{21} & 0 & 0 & \cdots & 0 \\ a_{31} & a_{32} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & 0 \end{pmatrix}
\qquad
\left[\overline{U}\right]=\begin{pmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & 0 & a_{23} & \cdots & a_{2n} \\ 0 & 0 & 0 & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}
$$

$$
\left[\overline{D}\right]=\begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix}
$$

$$[M] = [\tilde{L}][\tilde{U}] = [\overline{L} + \overline{D}][\overline{D}^{-1}][\overline{D} + \overline{U}] = [\overline{L}\,\overline{D}^{-1} + I][\overline{D} + \overline{U}]$$
$$= [\overline{L} + \overline{D}][I + \overline{D}^{-1}\overline{U}]$$

$$[\overline{L}] + [\overline{D}] = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ a_{21} & 0 & 0 & \cdots & 0 \\ a_{31} & a_{32} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & 0 \end{pmatrix} + \begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix} = [\tilde{L}]$$

$$[I] + [\overline{D}^{-1}\overline{U}] = \begin{pmatrix} 1 & a_{12}/a_{11} & a_{13}/a_{11} & \cdots & a_{1n}/a_{11} \\ 0 & 1 & a_{23}/a_{22} & \cdots & a_{2n}/a_{22} \\ 0 & 0 & 1 & \cdots & a_{3n}/a_{33} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} = [\tilde{U}]$$

Forward/Backward Subst. in LU-GS

$$[M]=[\tilde{L}][\tilde{U}]=[\bar{L}+\bar{D}][\bar{D}^{-1}][\bar{D}+\bar{U}]=[\bar{L}\bar{D}^{-1}+I][\bar{D}+\bar{U}]=[\bar{L}+\bar{D}][I+\bar{D}^{-1}\bar{U}]$$

Forward Substitution

$$[\bar{L}+\bar{D}]\{y\}=\{r\}\Rightarrow\{y\}=[\bar{D}^{-1}](\{r\}-[\bar{L}]\{y\})\Rightarrow y_i=\bar{D}_{ii}^{-1}\left(r_i-\sum_{j=1}^{i-1}\bar{L}_{ij}y_j\right)$$

Backward Substitution

$$[I+\bar{D}^{-1}\bar{U}]\{z\}=\{y\}\Rightarrow\{z\}=\{y\}-[\bar{D}^{-1}][\bar{U}]\{z\}\Rightarrow z_i=y_i-\bar{D}_{ii}^{-1}\left[\sum_{j=i+1}^{N}\bar{U}_{ij}z_j\right]$$

$$[\bar{L}]+[\bar{D}]=\begin{pmatrix}0&0&0&\cdots&0\\a_{21}&0&0&\cdots&0\\a_{31}&a_{32}&0&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\a_{n1}&a_{n2}&a_{n3}&\cdots&0\end{pmatrix}+\begin{pmatrix}a_{11}&0&0&\cdots&0\\0&a_{22}&0&\cdots&0\\0&0&a_{33}&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&a_{nn}\end{pmatrix}=\begin{pmatrix}a_{11}&0&0&\cdots&0\\a_{21}&a_{22}&0&\cdots&0\\a_{31}&a_{32}&a_{33}&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\a_{n1}&a_{n2}&a_{n3}&\cdots&a_{nn}\end{pmatrix}=[\tilde{L}]$$

$$[I]+[\bar{D}^{-1}\bar{U}]=\begin{pmatrix}1&a_{12}/a_{11}&a_{13}/a_{11}&\cdots&a_{1n}/a_{11}\\0&1&a_{23}/a_{22}&\cdots&a_{2n}/a_{22}\\0&0&1&\cdots&a_{3n}/a_{33}\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&1\end{pmatrix}=[\tilde{U}]$$

Forward/Backward Subst. in LU-GS

$$\left[\tilde{L}\right]\{z\} = \{z\}$$

$$\left[\tilde{U}\right]\{z\} = \{z\}$$

```
!C
!C +----------------+
!C | {z}= [Minv]{r} |
!C +----------------+
!C===
      do i= 1, N
        W(i,Z)= W(i,R)
      enddo

      do i= 1, N
        WVAL= W(i,Z)
        do k= indexL(i-1)+1, indexL(i)
          WVAL= WVAL -  AL(k) * W(itemL(k),Z)
        enddo
        W(i,Z)= WVAL / D(i)
      enddo

      do i= N, 1, -1
        SW  = 0.0d0
        do k= indexU(i), indexU(i-1)+1, -1
          SW= SW + AU(k) * W(itemU(k),Z)
        enddo
        W(i,Z)= W(i,Z) - SW / D(i)
      enddo
!C===
```

$$z_i = \overline{D}_{ii}^{-1}\left(z_i - \sum_{j=1}^{i-1} \overline{L}_{ij} z_j\right)$$

$$\text{WVAL} = z_i - \sum_{j=1}^{i-1} \overline{L}_{ij} z_j$$

$$z_i = z_i - \overline{D}_{ii}^{-1}\left[\sum_{j=i+1}^{N} \overline{U}_{ij} z_j\right]$$

$$\text{SW} = \sum_{j=i+1}^{N} \overline{U}_{ij} z_j$$

Parallel Preconditioning Method using MPI

Localized SGS/SSOR Preconditioning

- SGS/SSOR: Global Operations (Forward/Backward Substitution)
  - NOT suitable for parallel computing
- Ignoring effects of external points for preconditioning
  - Block-Jacobi Localized Preconditioning
- WEAKER than original SGS/SSOR
  - More PE's, more iterations

$$(L)\{z\}=\{r\}$$

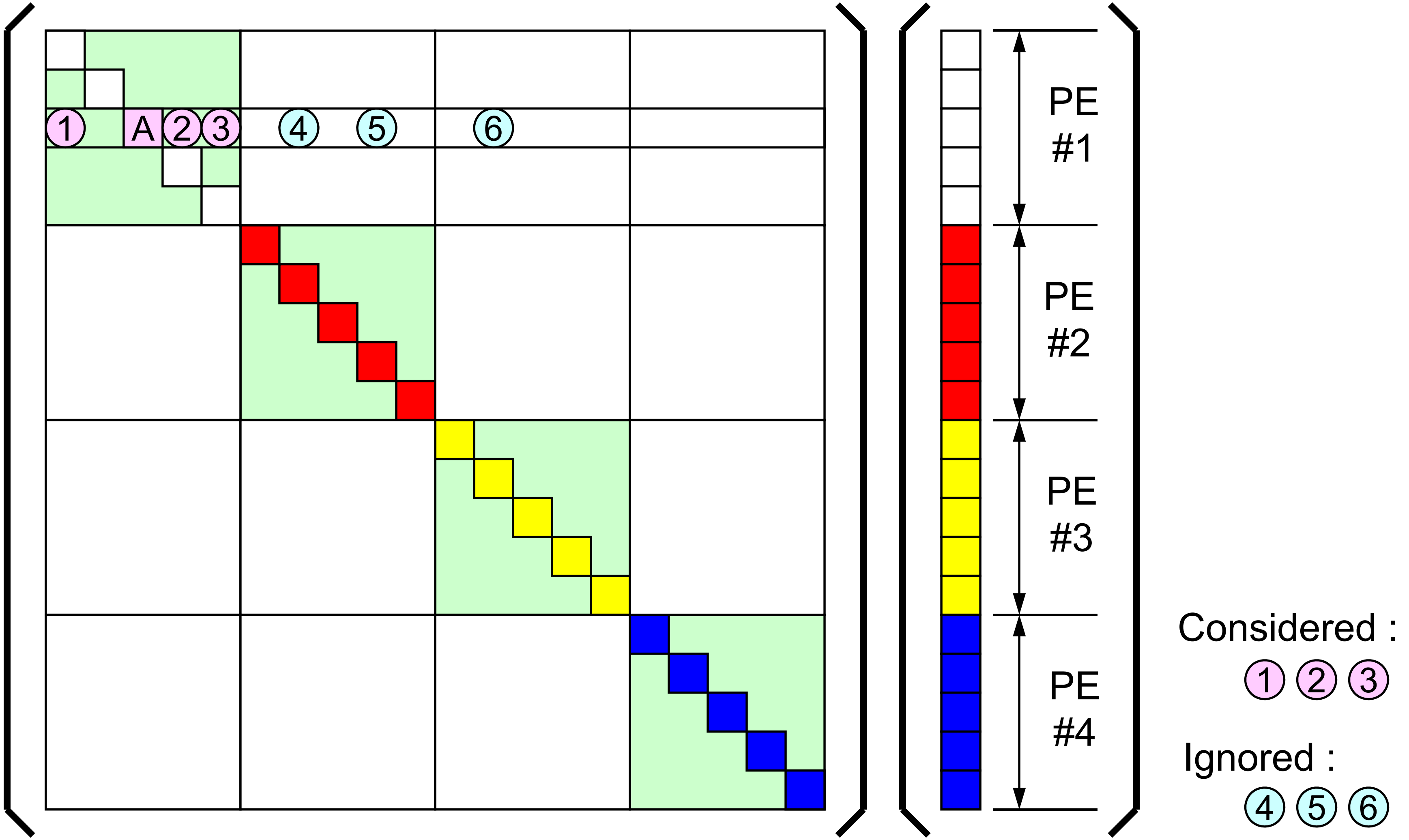
$$(U)\{z\}=\{z\}$$

```
!C
!C +----------------+
!C | {z}= [Minv]{r} |
!C +----------------+
!C===
      do i= 1, N
        W(i,Z)= W(i,R)
      enddo

      do i= 1, N
        WVAL= W(i,Z)
        do k= indexL(i-1)+1, indexL(i)
          WVAL= WVAL - AL(k) * W(itemL(k),Z)
        enddo
        W(i,Z)= WVAL / D(i)
      enddo

      do i= N, 1, -1
        SW  = 0.0d0
        do k= indexU(i), indexU(i-1)+1, -1
          SW= SW + AU(k) * W(itemU(k),Z)
        enddo
        W(i,Z)= W(i,Z) - SW / D(i)
      enddo
!C===
```

Localized SGS/SSOR Preconditioning

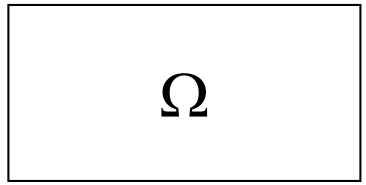


Overlapped Additive Schwartz Domain Decomposition Method

Stabilization of Localized Preconditioning: ASDD

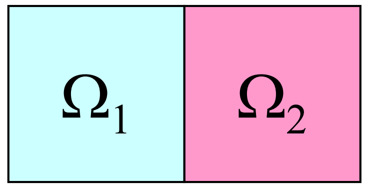
<u>Global Operation</u>

$$Mz = r$$



<u>Local Operation</u>

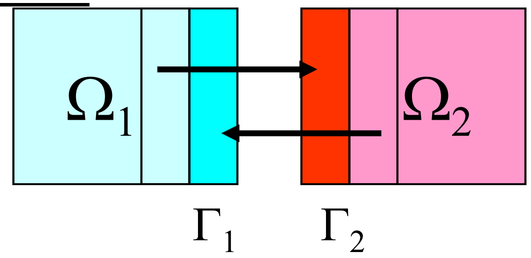
$$z_{\Omega_1} = M_{\Omega_1}^{-1} r_{\Omega_1}, \quad z_{\Omega_2} = M_{\Omega_2}^{-1} r_{\Omega_2}$$



<u>Global Nesting Correction: Repeating -> Stable</u>

$$z_{\Omega_1}^{n} = z_{\Omega_1}^{n-1} + M_{\Omega_1}^{-1}\left(r_{\Omega_1} - M_{\Omega_1} z_{\Omega_1}^{n-1} - M_{\Gamma_1} z_{\Gamma_1}^{n-1}\right)$$

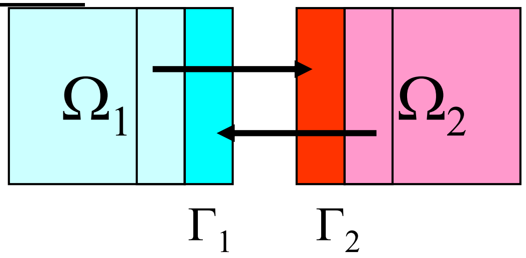
$$z_{\Omega_2}^{n} = z_{\Omega_2}^{n-1} + M_{\Omega_2}^{-1}\left(r_{\Omega_2} - M_{\Omega_2} z_{\Omega_2}^{n-1} - M_{\Gamma_2} z_{\Gamma_2}^{n-1}\right)$$



Overlapped Additive Schwartz Domain Decomposition Method

Stabilization of Localized Preconditioning: ASDD

<u>Global Nesting Correction: Repeating -> Stable</u>



$$z_{\Omega_1}^{n} = z_{\Omega_1}^{n-1} + M_{\Omega_1}^{-1}(r_{\Omega_1} - M_{\Omega_1} z_{\Omega_1}^{n-1} - M_{\Gamma_1} z_{\Gamma_1}^{n-1})$$

$$z_{\Omega_2}^{n} = z_{\Omega_2}^{n-1} + M_{\Omega_2}^{-1}(r_{\Omega_2} - M_{\Omega_2} z_{\Omega_2}^{n-1} - M_{\Gamma_2} z_{\Gamma_2}^{n-1})$$

$$\Delta r_{\Omega_1} = r_{\Omega_1} - M_{\Omega_1} z_{\Omega_1}^{n-1} - M_{\Gamma_1} z_{\Gamma_1}^{n-1}$$

$$\Delta z_{\Omega_1} = M_{\Omega_1}^{-1} \Delta r_{\Omega_1} \quad where \quad \Delta z_{\Omega_1} = z_{\Omega_1}^{n} - z_{\Omega_1}^{n-1}$$

$$z_{\Omega_1}^{n} = z_{\Omega_1}^{n-1} + \Delta z_{\Omega_1} = z_{\Omega_1}^{n-1} + M_{\Omega_1}^{-1} \Delta r_{\Omega_1} = z_{\Omega_1}^{n-1} + M_{\Omega_1}^{-1}(r_{\Omega_1} - M_{\Omega_1} z_{\Omega_1}^{n-1} - M_{\Gamma_1} z_{\Gamma_1}^{n-1})$$

Overlapped Additive Schwartz Domain Decomposition Method

Effect of additive Schwartz domain decomposition for solid mechanics example example with 3×44^3 DOF on Hitachi SR2201, Number of ASDD cycle/iteration= 1, $\varepsilon = 10^{-8}$

	NO Additive Schwartz			WITH Additive Schwartz		
PE #	Iter. #	Sec.	Speed Up	Iter.#	Sec.	Speed Up
1	204	233.7	-	144	325.6	-
2	253	143.6	1.63	144	163.1	1.99
4	259	74.3	3.15	145	82.4	3.95
8	264	36.8	6.36	146	39.7	8.21
16	262	17.4	13.52	144	18.7	17.33
32	268	9.6	24.24	147	10.2	31.80
64	274	6.6	35.68	150	6.5	50.07