Gaussian Elimination
ガウスの消去法

Linear Equations with "n" unknowns

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
$$

$$\Updownarrow$$

Matrix Form

$$
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\
\vdots & & \vdots & & \vdots \\
\vdots & & & & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn} & b_n
\end{pmatrix}
$$

Gaussian Elimination
ガウスの消去法

- Gaussian Elimination
  - Forward Elimination　　　前進消去
  - Backward Substitution　　後退代入

Gaussian Elimination (cont.)

Following equations are obtained though transformations which do not change solutions.

$$
\begin{aligned}
x_1 + a'_{12}x_2 + \cdots + a'_{1n}x_n &= b'_1 \\
x_2 + \cdots \quad + a'_{2n}x_n &= b'_2 \\
&\vdots \\
x_n &= b'_n
\end{aligned}
$$

$\Updownarrow$

Matrix Form

$$
\begin{pmatrix}
1 & a'_{12} & \cdots & a'_{1n} & b'_1 \\
0 & 1 & \cdots & a'_{2n} & b'_2 \\
\vdots & 0 & \vdots & & \vdots \\
\vdots & \vdots & & \vdots & \vdots \\
0 & 0 & \cdots & 1 & b'_n
\end{pmatrix}
$$

Transformations which do not change solutions of target equations

- Multiplication by Scalar
  - Multiply a row by a scalar value
- Add (Mult. by Scalar) to other row
  - Multiply a row by a scalar value, and add them to other row
- Exchange Order of Equations
  - Exchange rows
- Exchange of Variables
  - Exchange columns

Gaussian Elimination

$$
\begin{aligned}
x_1 + a'_{12}x_2 + \cdots + a'_{1n}x_n &= b'_1 \\
x_2 + \cdots \quad + a'_{2n}x_n &= b'_2 \\
&\vdots \\
x_n &= b'_n
\end{aligned}
\qquad (2)
$$

Solution of these Eqn's ?

Solutions are obtained through row-by-row substitution.

$$
\begin{aligned}
x_n &= b'_n \\
x_{n-1} &= b'_{n-1} - a'_{n-1,n}x_n \\
&\vdots \\
x_1 &= b'_1 - (a'_{12}x_2 + \cdots + a'_{1n}x_n)
\end{aligned}
$$

Forward Elimination

Forward elimination

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
x_1 + a'_{12}x_2 + \cdots + a'_{1n}x_n &= b'_1 \\
x_2 + \cdots \quad + a'_{2n}x_n &= b'_2 \\
&\vdots \\
x_n &= b'_n
\end{aligned}
$$

Transform equations in this manner

Backward Substitution

$$
\begin{aligned}
x_1 + a'_{12}x_2 + \cdots + a'_{1n}x_n &= b'_1 \\
x_2 + \cdots \quad + a'_{2n}x_n &= b'_2 \\
&\vdots \\
x_n &= b'_n
\end{aligned}
$$

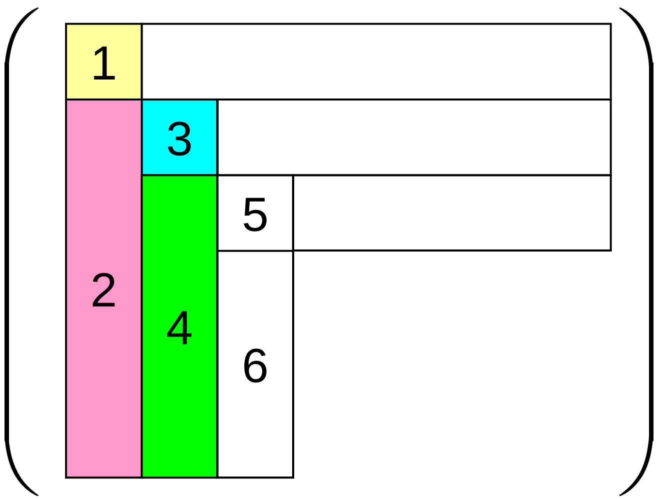
Backward substitution

$$
\begin{aligned}
x_n &= b'_n \\
x_{n-1} &= b'_{n-1} - a'_{n-1,n}x_n \\
&\vdots \\
x_1 &= b'_1 - (a'_{12}x_2 + \cdots + a'_{1n}x_n)
\end{aligned}
$$

Obtain solutions using transformed equations

Forward Elimination in Mat. Form

$$
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\
\vdots & & \vdots & & \vdots \\
\vdots & & & & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn} & b_n
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
1 & a'_{12} & \cdots & a'_{1n} & b'_1 \\
0 & 1 & \cdots & a'_{2n} & b'_2 \\
\vdots & 0 & \vdots & & \vdots \\
\vdots & \vdots & & \vdots & \vdots \\
0 & 0 & \cdots & 1 & b'_n
\end{pmatrix}
$$



- 1: $a_{11} \rightarrow 1$
- 2: $a_{21}, a_{31}, \cdots, a_{n1} \rightarrow 0$
- 3: $a_{22} \rightarrow 1$
- 4: $a_{32}, a_{42}, \cdots, a_{n2} \rightarrow 0$

Selecting of Pivot：Pivot選択
(for avoiding inaccurate solution)

$$\begin{pmatrix} 1 & a_{12} & \cdots & a_{1k} & \cdots & a_{1n} & b_1 \\ 0 & 1 & \cdots & a_{2k} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & \boxed{a_{kk}} & \cdots & a_{kn} & b_k \\ 0 & 0 & \cdots & a_{k+1,k} & \cdots & a_{k+1,n} & b_{k+1} \\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{nk} & \cdots & a_{nn} & b_n \end{pmatrix}$$

Pivot

$$p = a_{kk}$$

Elimination of these Components:

$$\frac{a_{kj}}{p} \to a_{kj} \qquad \frac{b_k}{p} \to b_k$$

$$\begin{cases} a_{ij} - a_{ik} a_{kj} \to a_{ij} \\ b_i - a_{ik} b_k \to b_i \end{cases}$$

Selection of Pivot, Effect on Errors

$|p| = 0$ → Cannot be divided by “p”

$|p|$ is small → $|a_{kj} / p|$ $|b_k / p|$ are large

→ $\begin{cases} a_{ij} - a_{ik} a_{kj} \to a_{ij} \\ b_i - a_{ik} b_k \to b_i \end{cases}$ Round-off error may occur

Anyway, large $|p|$ is preferable !!

Strategy

While selecting larger “pivots” for Gaussian Elimination, more accurate solutions may be obtained.

Partial Pivoting：部分Pivoting

Pivoting: Selection of “Pivot”

$$
\begin{pmatrix}
1 & a_{12} & \cdots & a_{1k} & \cdots & a_{1n} & b_1 \\
0 & 1 & \cdots & a_{2k} & \cdots & a_{2n} & b_2 \\
\vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & \cdots & a_{kk} & \cdots & a_{kn} & b_k \\
\vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & \cdots & a_{Ik} & \cdots & a_{In} & b_I \\
\vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & \cdots & a_{nk} & \cdots & a_{nn} & b_n
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
1 & a_{12} & \cdots & a_{1k} & \cdots & a_{1n} & b_1 \\
0 & 1 & \cdots & a_{2k} & \cdots & a_{2n} & b_2 \\
\vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & \cdots & a_{Ik} & \cdots & a_{In} & b_I \\
\vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & \cdots & a_{kk} & \cdots & a_{kn} & b_k \\
\vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & \cdots & a_{nk} & \cdots & a_{nn} & b_n
\end{pmatrix}
$$

Select “I-th row” which provides the largest value of a_{ik} and exchange “I-th row” with “k-th row”

Continue same process (solution is not changed)

Full Pivoting：完全Pivoting

$$
\begin{pmatrix}
1 & a_{12} & \cdots & a_{1k} & \cdots & a_{1J} & \cdots & a_{1n} & b_1 \\
0 & 1 & \cdots & a_{2k} & \cdots & a_{2J} & \cdots & a_{2n} & b_2 \\
\vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & \cdots & a_{kk} & \cdots & a_{kJ} & \cdots & a_{kn} & b_k \\
\vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & \cdots & a_{Ik} & \cdots & a_{IJ} & \cdots & a_{In} & b_I \\
\vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & \cdots & a_{nk} & \cdots & a_{nJ} & \cdots & a_{nn} & b_n
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
1 & a_{12} & \cdots & a_{1k} & \cdots & a_{1J} & \cdots & a_{1n} & b_1 \\
0 & 1 & \cdots & a_{2k} & \cdots & a_{2J} & \cdots & a_{2n} & b_2 \\
\vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & \cdots & a_{IJ} & \cdots & a_{Ik} & \cdots & a_{In} & b_I \\
\vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & \cdots & a_{kJ} & \cdots & a_{kk} & \cdots & a_{kn} & b_k \\
\vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & \cdots & a_{nJ} & \cdots & a_{nk} & \cdots & a_{nn} & b_n
\end{pmatrix}
$$

Select “I-th row” and “J-th” column which provides the largest value of a_{ik}, and xchnage “I-th row” with “k-th row”, and “J-th col.” with “k-th col.”, respectively

Continue same process (solution is not changed)

If “column” is exchanged, “New-to-Old” relationship must be saved.

Example: Gaussian Elimination with Partial/Full Pivoting

$$\begin{bmatrix} 2 & 4 & -2 \\ 1 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 8 \\ 6 \\ 9 \end{Bmatrix}$$

Solve this equation

Forward Elimination

Extended Matrix Form

$$\left(\begin{array}{cccc} 2 & 4 & -2 & 8 \\ 1 & 2 & 1 & 6 \\ 1 & 3 & 2 & 9 \\ \hline 1 & 2 & 3 & \end{array}\right)$$

Number corresponding to column ID
(not changed if columns are not exchanged)

Partial Pivoting (1/2)

Starting Elimination

$$\begin{pmatrix} \boxed{2} & 4 & -2 & 8 \\ 1 & 2 & 1 & 6 \\ 1 & 3 & 2 & 9 \end{pmatrix}$$

1. 1st Row÷2

$$\begin{pmatrix} 1 & 2 & -1 & 4 \\ 1 & 2 & 1 & 6 \\ 1 & 3 & 2 & 9 \end{pmatrix}$$

2. 2nd Row−1st Row, 3rd Row−1st Row

$$\begin{pmatrix} 1 & 2 & -1 & 4 \\ 0 & \boxed{0} & 2 & 2 \\ 0 & 1 & 3 & 5 \end{pmatrix} \longrightarrow$$

3. 2nd Row⇔3rd Row (Exchange)

$$\begin{pmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 2 \end{pmatrix}$$

Partial Pivoting (2/2)

3. 2nd Row⇔3rd Row (Exchange)

$$\begin{pmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & \boxed{2} & 2 \end{pmatrix}$$

4. 3rd Row÷2

$$\begin{pmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 4 \\ 5 \\ 1 \end{Bmatrix}$$

Backward Substitution

$$x_3 = 1$$
$$x_2 = 5 - 3x_3 = 2$$
$$x_1 = 4 + x_2 - 2x_3 = 1$$

Full Pivoting (1/5)

$$\begin{pmatrix} 2 & \boxed{4} & -2 & 8 \\ 1 & 2 & 1 & 6 \\ 1 & 3 & 2 & 9 \\ \hline 1 & 2 & 3 & \end{pmatrix} \xrightarrow{1^{st}\ R \Leftrightarrow 2^{nd}\ R} \begin{pmatrix} 4 & 2 & -2 & 8 \\ 2 & 1 & 1 & 6 \\ 3 & 1 & 2 & 9 \\ \hline 2 & 1 & 3 & \end{pmatrix}$$

Find “pivot” with largest value

$$\Big\downarrow\ 1^{st}\ R \times (1/4)$$

$$\begin{pmatrix} 1 & 1/2 & -1/2 & 2 \\ 2 & 1 & 1 & 6 \\ 3 & 1 & 2 & 9 \\ \hline 2 & 1 & 3 & \end{pmatrix}$$

Full Pivoting (2/5)

$$\left(\begin{array}{cccc} 1 & 1/2 & -1/2 & 2 \\ 2 & 1 & 1 & 6 \\ 3 & 1 & 2 & 9 \\ \hline 2 & 1 & 3 & \end{array}\right) \xrightarrow[\text{3rd R}-\text{1st R}\times 3]{\text{2nd R}-\text{1st R}\times 2} \left(\begin{array}{cccc} 1 & 1/2 & -1/2 & 2 \\ 0 & 0 & 2 & 2 \\ 0 & -1/2 & \boxed{7/2} & 3 \\ \hline 2 & 1 & 3 & \end{array}\right)$$

Find “pivot” with largest value

2nd R⇔3rd R
2nd C⇔3rd C

$$\left(\begin{array}{cccc} 1 & -1/2 & 1/2 & 2 \\ 0 & 7/2 & -1/2 & 3 \\ 0 & 2 & 0 & 2 \\ \hline 2 & 3 & 1 & \end{array}\right)$$

Full Pivoting (3/5)

$$\left(\begin{array}{cccc} 1 & -1/2 & 1/2 & 2 \\ 0 & 7/2 & -1/2 & 3 \\ 0 & 2 & 0 & 2 \\ \hline 2 & 3 & 1 & \end{array}\right) \xrightarrow{2^{nd}\ R \times (2/7)} \left(\begin{array}{cccc} 1 & -1/2 & 1/2 & 2 \\ 0 & 1 & -1/7 & 6/7 \\ 0 & 2 & 0 & 2 \\ \hline 2 & 3 & 1 & \end{array}\right)$$

$$\xrightarrow{3^{rd}\ R - 2^{nd}\ R \times 2} \left(\begin{array}{cccc} 1 & -1/2 & 1/2 & 2 \\ 0 & 1 & -1/7 & 6/7 \\ 0 & 0 & 2/7 & 2/7 \\ \hline 2 & 3 & 1 & \end{array}\right) \xrightarrow{3^{rd}\ R \times (7/2)} \left(\begin{array}{cccc} 1 & -1/2 & 1/2 & 2 \\ 0 & 1 & -1/7 & 6/7 \\ 0 & 0 & 1 & 1 \\ \hline 2 & 3 & 1 & \end{array}\right)$$

Full Pivoting (4/5)

Finally,

$$\left(\begin{array}{cccc} 2 & 4 & -2 & 8 \\ 1 & 2 & 1 & 6 \\ 1 & 3 & 2 & 2 \\ \hline 1 & 2 & 3 & \end{array}\right) \xrightarrow{\text{Forward Elimination}} \left(\begin{array}{cccc} 1 & -1/2 & 1/2 & 2 \\ 0 & 1 & -1/7 & 6/7 \\ 0 & 0 & 1 & 1 \\ \hline 2 & 3 & 1 & \end{array}\right)$$

Forward Elimination Completed

$$\left(\begin{array}{cccc} 1 & -1/2 & 1/2 & 2 \\ 0 & 1 & -1/7 & 6/7 \\ 0 & 0 & 1 & 1 \\ \hline 2 & 3 & 1 & \end{array}\right) \xrightarrow{\text{Backward Substitution}}$$

$$x_3' = 1$$

$$x_2' = \frac{6}{7} + \frac{1}{7} x_3' = 1$$

$$x_1' = 2 + \frac{1}{2} \times x_2' - \frac{1}{2} \times x_3' = 2$$

Full Pivoting (5/5)

Re-exchange columns, according to original numbering:

$$x_1' \Leftrightarrow x_2$$

$$x_2' \Leftrightarrow x_3$$

$$x_3' \Leftrightarrow x_1$$

Finally, following solutions are obtained:

$$\begin{cases} x_1 = 1 \\ x_2 = 2 \\ x_3 = 1 \end{cases}$$