

Proof (2/3)

Mathematical Induction

数学的帰納法

$$\begin{aligned} (r^{(i)}, r^{(j)}) &= 0 \quad (i \neq j) \\ (p^{(i)}, Ap^{(j)}) &= 0 \quad (i \neq j) \end{aligned} \quad (*)$$

(*) is satisfied for $i \leq k, j \leq k$ where $i \neq j$ または $0 \leq i < j \leq k$

$$\begin{aligned} \text{if } i < k \quad (r^{(k+1)}, r^{(i)}) &= (r^{(i)}, r^{(k+1)}) \stackrel{(2)}{=} (r^{(i)}, r^{(k)} - \alpha_k Ap^{(k)}) \\ &\stackrel{(*)}{=} -\alpha_k (r^{(i)}, Ap^{(k)}) \stackrel{(4)}{=} -\alpha_k (p^{(i)} - \beta_{i-1} p^{(i-1)}, Ap^{(k)}) \\ &= -\alpha_k (p^{(i)}, Ap^{(k)}) + \alpha_k \beta_{i-1} (p^{(i-1)}, Ap^{(k)}) \stackrel{(*)}{=} 0 \end{aligned}$$

$$\begin{aligned} \text{if } i = k \quad (r^{(k+1)}, r^{(k)}) &\stackrel{(2)}{=} (r^{(k)}, r^{(k)}) - (r^{(k)}, \alpha_k Ap^{(k)}) \\ &\stackrel{(3)}{=} (r^{(k)}, r^{(k)}) - (p^{(k)} - \beta_{k-1} p^{(k-1)}, \alpha_k Ap^{(k)}) \\ &\stackrel{(*)}{=} (r^{(k)}, r^{(k)}) - \alpha_k (p^{(k)}, Ap^{(k)}) \stackrel{(1)}{=} (r^{(k)}, r^{(k)}) - (p^{(k)}, r^{(k)}) \\ &\stackrel{(3)}{=} (r^{(k)}, r^{(k)}) - (\beta_{k-1} p^{(k-1)} + r^{(k)}, r^{(k)}) \\ &= -\beta_{k-1} (p^{(k-1)}, r^{(k)}) \stackrel{(2)}{=} -\beta_{k-1} (p^{(k-1)}, r^{(k-1)} - \alpha_{k-1} Ap^{(k-1)}) \\ &= -\beta_{k-1} \left\{ (p^{(k-1)}, r^{(k-1)}) - \alpha_{k-1} (p^{(k-1)}, Ap^{(k-1)}) \right\} \stackrel{(1)}{=} 0 \end{aligned}$$

$$(1) \alpha_k = \frac{(p^{(k)}, r^{(k)})}{(p^{(k)}, Ap^{(k)})}$$

$$(2) r^{(k+1)} = r^{(k)} - \alpha_k Ap^{(k)}$$

$$(3) p^{(k+1)} = r^{(k+1)} + \beta_k p^{(k)}$$

$$(4) \beta_k = \frac{-(r^{(k+1)}, Ap^{(k)})}{(p^{(k)}, Ap^{(k)})}$$

Proof (3/3)

Mathematical Induction

数学的帰納法

$$\begin{aligned} (r^{(i)}, r^{(j)}) &= 0 \quad (i \neq j) \\ (p^{(i)}, Ap^{(j)}) &= 0 \quad (i \neq j) \end{aligned} \quad (*)$$

(*) is satisfied for $i \leq k, j \leq k$ where $i \neq j$ または $0 \leq i < j \leq k$

$$\begin{aligned} \text{if } i < k \quad (p^{(k+1)}, Ap^{(i)}) &\stackrel{(3)}{=} (r^{(k+1)} + \beta_k p^{(k)}, Ap^{(i)}) \\ &\stackrel{(*)}{=} (r^{(k+1)}, Ap^{(i)}) \end{aligned}$$

$$\stackrel{(2)}{=} \frac{1}{\alpha_k} (r^{(k+1)}, r^{(i)} - r^{(i-1)}) = 0$$

$$\begin{aligned} \text{if } i = k \quad (p^{(k+1)}, Ap^{(k)}) &\stackrel{(3)}{=} (r^{(k+1)}, Ap^{(k)}) + \beta_k (p^{(k)}, Ap^{(k)}) \\ &\stackrel{(4)}{=} 0 \end{aligned}$$

$$(1) \alpha_k = \frac{(p^{(k)}, r^{(k)})}{(p^{(k)}, Ap^{(k)})}$$

$$(2) r^{(k+1)} = r^{(k)} - \alpha_k Ap^{(k)}$$

$$(3) p^{(k+1)} = r^{(k+1)} + \beta_k p^{(k)}$$

$$(4) \beta_k = \frac{-(r^{(k+1)}, Ap^{(k)})}{(p^{(k)}, Ap^{(k)})}$$

$$\begin{aligned} (p^{(k)}, r^{(k)}) &\stackrel{(3)}{=} (r^{(k)} + \beta_{k-1} p^{(k-1)}, r^{(k)}) \\ &= (\beta_{k-1} p^{(k-1)}, r^{(k)}) + (r^{(k)}, r^{(k)}) = (r^{(k)}, r^{(k)}) \end{aligned}$$

$$(1) \alpha_k = \frac{(p^{(k)}, r^{(k)})}{(p^{(k)}, Ap^{(k)})}$$

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$$\begin{aligned} \because (Ap^{(k)}, y - x^{(k+1)}) &= (p^{(k)}, Ay - Ax^{(k+1)}) \\ &= (p^{(k)}, b - Ax^{(k+1)}) = (p^{(k)}, r^{(k+1)}) = 0 \end{aligned}$$

$$(p^{(i)}, r^{(k+1)}) = 0, \quad i = 0, 1, \dots, k$$

$$x^{(k+1)} = x^{(i+1)} + \sum_{j=i+1}^k \alpha_j p^{(j)}$$

$$\begin{aligned} r^{(k+1)} &= b - Ax^{(k+1)} = b - A \left[x^{(i+1)} + \sum_{j=i+1}^k \alpha_j p^{(j)} \right] \\ &= [b - Ax^{(i+1)}] - \sum_{j=i+1}^k \alpha_j Ap^{(j)} = r^{(i+1)} - \sum_{j=i+1}^k \alpha_j Ap^{(j)} \end{aligned}$$

$$\begin{aligned} (p^{(i)}, r^{(k+1)}) &= \left(p^{(i)}, r^{(i+1)} - \sum_{j=i+1}^k \alpha_j Ap^{(j)} \right) \\ &= \underbrace{(p^{(i)}, r^{(i+1)})}_{=0} - \underbrace{\left(p^{(i)}, \sum_{j=i+1}^k \alpha_j Ap^{(j)} \right)}_{=0} = 0 \end{aligned}$$