Introduction to FEM

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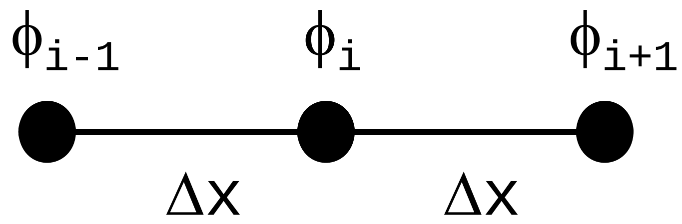
FDM and FEM

- Numerical Method for solving PDE's
  - Space is discretized into small pieces (elements, meshes)
- Finite Difference Method (FDM)
  - Differential derivatives are directly approximated using Taylor Series Expansion.
- Finite Element Method（FEM）
  - Solving "weak form" derived from integral equations.
    - "Weak solutions" are obtained.
  - Method of Weighted Residual (MWR), Variational Method
  - Suitable for Complicated Geometries
    - Although FDM can handle complicated geometries ...

Finite Difference Method (FDM) Taylor Series Expansion



2nd-Order Central Difference

$$\phi_{i+1} = \phi_i + \Delta x \left(\frac{\partial \phi}{\partial x} \right)_i + \frac{(\Delta x)^2}{2!} \left(\frac{\partial^2 \phi}{\partial x^2} \right)_i + \frac{(\Delta x)^3}{3!} \left(\frac{\partial^3 \phi}{\partial x^3} \right)_i \cdots$$

$$\phi_{i-1} = \phi_i - \Delta x \left(\frac{\partial \phi}{\partial x} \right)_i + \frac{(\Delta x)^2}{2!} \left(\frac{\partial^2 \phi}{\partial x^2} \right)_i - \frac{(\Delta x)^3}{3!} \left(\frac{\partial^3 \phi}{\partial x^3} \right)_i \cdots$$

$$\frac{\phi_{i+1} - \phi_{i-1}}{2\Delta x} = \left(\frac{\partial \phi}{\partial x} \right)_i + \frac{2 \times (\Delta x)^2}{3!} \left(\frac{\partial^3 \phi}{\partial x^3} \right)_i \cdots$$

1D Heat Conduction

- 2^{nd}-Order Central Difference

$$\left(\frac{d^2\phi}{dx^2}\right)_i \approx \frac{\left(\frac{d\phi}{dx}\right)_{i+1/2} - \left(\frac{d\phi}{dx}\right)_{i-1/2}}{\Delta x} = \frac{\frac{\phi_{i+1}-\phi_i}{\Delta x} - \frac{\phi_i - \phi_{i-1}}{\Delta x}}{\Delta x} = \frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{\Delta x^2}$$

- Linear Equation at Each Grid Point

$$\frac{d^2\phi}{dx^2} + BF = 0 \quad \longrightarrow \quad \frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{\Delta x^2} + BF(i) = 0 \quad (1 \le i \le N)$$

$$\frac{1}{\Delta x^2}\phi_{i+1} - \frac{2}{\Delta x^2}\phi_i + \frac{1}{\Delta x^2}\phi_{i-1} + BF(i) = 0 \quad (1 \le i \le N)$$

$$A_L(i) \times \phi_{i-1} + A_D(i) \times \phi_i + A_R(i) \times \phi_{i+1} = BF(i) \quad (1 \le i \le N)$$

$$A_L(i) = \frac{1}{\Delta x^2},\ A_D(i) = -\frac{2}{\Delta x^2},\ A_R(i) = \frac{1}{\Delta x^2}$$

FDM can handle complicated geometries: BFC

Handbook of Grid Generation

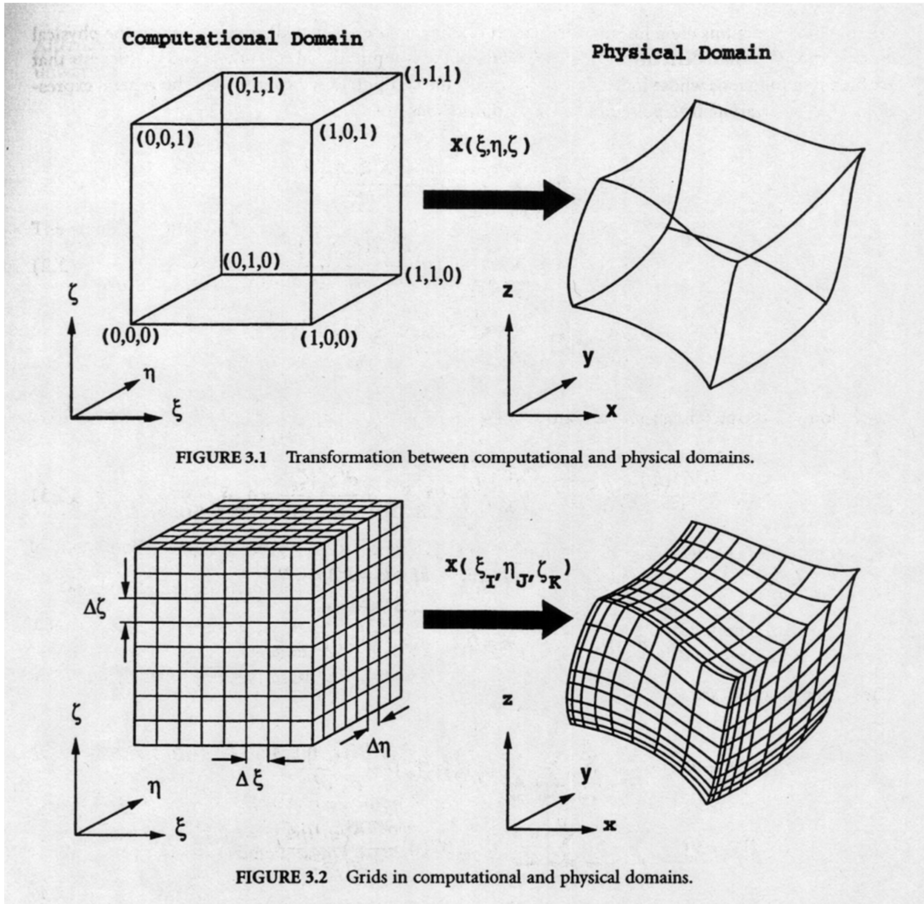
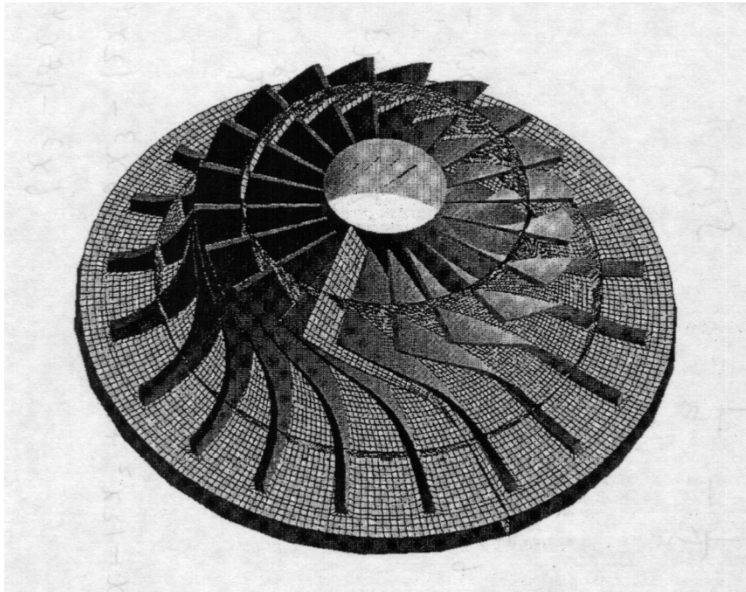


FIGURE 3.1 Transformation between computational and physical domains.

FIGURE 3.2 Grids in computational and physical domains.



History of FEM

- In 1950's, FEM was originally developed as a method for structure analysis of wings of airplanes under collaboration between Boeing and University of Washington (M.J. Turner, H.C. Martin etc.).
  - "Beam Theory" cannot be applied to sweptback wings for airplanes with jet engines.
- Extended to Various Applications
  - Non-Linear: T.J.Oden
  - Non-Structure Mechanics: O.C.Zienkiewicz
- Commercial Package
  - NASTRAN
    - Originally developed by NASA
    - Commercial Version by MSC
    - PC version is widely used in industries

Recent Research Topics

- Non-Linear Problems
  - Crash, Contact, Non-Linear Material
  - Discontinuous Approach
    - X-FEM
- Parallel Computing
  - also in commercial codes
- Adaptive Mesh Refinement (AMR)
  - Shock Wave, Separation
  - Stress Concentration
  - Dynamic Load Balancing (DLB) at Parallel Computing
- Mesh Generation
  - Large-Scale Parallel Mesh Generation

- Numerical Method for PDE (Method of Weighted Residual)
- Gauss-Green’s Theorem
- Numerical Method for PDE (Variational Method)

Approximation Method for PDE

Partial Differential Equations: 偏微分方程式

- Consider solving the following differential equation (boundary value problem), domain V, boundary S :

$$L(u) = f$$

- u (solution of the equation) can be approximated by function u_M (linear combination)

$$u_M = \sum_{i=1}^{M} a_i \Psi_i$$

Ψ_i Trial/Test Function（試行関数）(known function of position, defined in domain and at boundary. “Basis” in linear algebra.

a_i Coefficients (unknown)

Method of Weighted Residual MWR: 重み付き残差法

- u_M is exact solution of u if R (residual：残差)= 0:

$$R = L(u_M) - f$$

- In MWR, consider the condition where the following integration of R multiplied by w (weight/weighting function：重み関数) over entire domain is 0

$$\int_V w R(u_M)\, dV = 0$$

- MWR provides “smoothed” approximate solution, which satisfies $R=0$ in the domain V

Variational Method (Ritz) (1/2) 変分法

- It is widely known that exact solution u provides extreme values (max/min) of "functional：汎関数" $I(u)$
  - Euler equation: differential equation satisfied by u, if functional has extreme values（極値）
  - Euler equation is satisfied, if u provides extreme values of $I(u)$.
  - *provide extreme values*：停留させる（or *stationarize*）
- For example, functional, which corresponds to governing equations of linear elasticity (principle of virtual work, equilibrium equations), is "principle of minimum potential energy (principle of minimum strain energy)" .

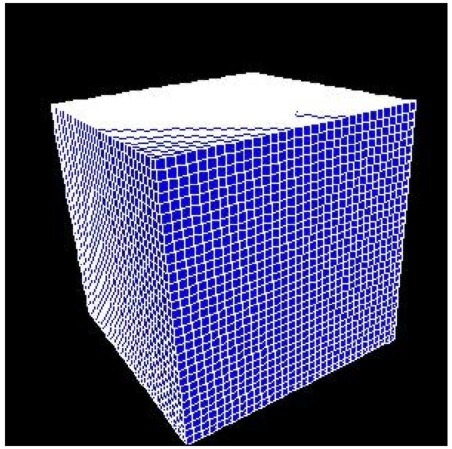
Variational Method (Ritz) (2/2) 変分法

- Substitute the following approx. solution into $I(u)$, and calculate coefficients a_i under the condition where $I_M=I(u_M)$ provides extreme values, then u_M is obtained:

$$u_M = \sum_{i=1}^{M} a_i \Psi_i$$

- Variational method is theoretical method, and can be only applied to differential equations, which has equivalent variational problem.
  - In this class, we mainly use MWR
  - Brief overview of Ritz method will given later today.

Finite Element Method (FEM) 有限要素法



- Entire region is discretized into fine elements（要素）, and the following approximation is applied to each element:

$$u_M = \sum_{i=1}^{M} a_i \Psi_i$$

- MWR or Variational Method is applied to each element
- Each element matrix is accumulated to global matrix, and solution of obtained linear equations provides approx. solution of PDE.
- Details of FEM will be provided after next week

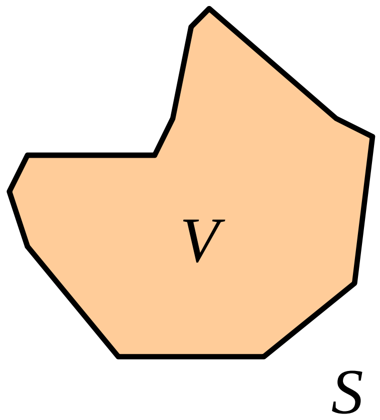
Example of MWR (1/3)

- Thermal Equation

$$\lambda\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+Q=0 \quad \text{in } V$$

λ: Conductivity, Q: Heat Gen./Volume

$T=0$ at boundary S



- Approximate Solution

$$T=\sum_{j=1}^{n} a_j \Psi_j$$

- Residual

$$R(a_j,x,y)=\lambda\sum_{j=1}^{n} a_j\left(\frac{\partial^2 \Psi_j}{\partial x^2}+\frac{\partial^2 \Psi_j}{\partial y^2}\right)+Q$$

Example of MWR (2/3)

- Multiply weighting function w_i, and apply integration over V:

$$\int_V w_i \, R \, dV = 0$$

- If a set of weighting function w_i is a set of n different functions, the above integration provides a set of n linear equations:
  - # trial/test functions = # weighting functions

$$\sum_{j=1}^{n} a_j \int_V w_i \, \lambda \left(\frac{\partial^2 \Psi_j}{\partial x^2} + \frac{\partial^2 \Psi_j}{\partial y^2} \right) dV = -\int_V w_i \, Q \, dV \quad (i = 1, \ldots, n)$$

Example of MWR (3/3)

- Matrix form of the equations is described as follows:

$$[B]\{a\} = \{Q\}$$

$$B_{ij} = \int_V w_i \, \lambda \left(\frac{\partial^2 \Psi_j}{\partial x^2} + \frac{\partial^2 \Psi_j}{\partial y^2} \right) dV, \quad Q_i = -\int_V w_i \, Q \, dV$$

Actual approach is slightly different from this (more detailed discussions after next week)

Various types of MWR's

- Various types of weighting functions

- Collocation Method 選点法
- Least Square Method 最小自乗法
- Galerkin Method ガラーキン法

Collocation Method

- Weighting function: Dirac's Delta Function δ

$$\delta(z) = \infty \quad if \quad z = 0$$

$$\delta(z) = 0 \quad if \quad z \neq 0, \quad \int_{-\infty}^{+\infty} \delta(z)\, dz = 1$$

$$w_i = \delta(\mathbf{x} - \mathbf{x_i}) \qquad \mathbf{x}: \text{location}$$

- In collocation method, R (residual) is set to 0 at n collocation points by feature of Dirac's Delta Fn. δ:

$$\int_V R\, \delta(\mathbf{x} - \mathbf{x_i})\, dV = R\,|_{\mathbf{x}=\mathbf{x_i}}$$

- If n increases, R approaches to 0 over entire domain.

Least Square Method

- Weighting function:

$$w_i = \frac{\partial R}{\partial a_i}$$

- Minimize the following integration according to a_i (unknowns):

$$I(a_i) = \int_V [R(a_i, \mathbf{x})]^2 \, dV$$

$$\frac{\partial}{\partial a_i}[I(a_i)] = 2\int_V \left[R(a_i, \mathbf{x}) \frac{\partial R(a_i, \mathbf{x})}{\partial a_i} \right] dV = 0$$

$$\Downarrow$$

$$\int_V \left[R(a_i, \mathbf{x}) \underline{\frac{\partial R(a_i, \mathbf{x})}{\partial a_i}} \right] dV = 0$$

Galerkin Method

- Weighting Function = Test/Trial Function:

$$w_i = \Psi_i$$

- Galerkin, Boris Grigorievich
  - 1871-1945
  - Engineer and Mathematician of Russia
  - He got a hint for Galerkin Method while he was imprisoned because of anti-czarism (1906-1907).

Example (1/2)

- Governing Equation

$$\frac{d^2u}{dx^2} + u + x = 0 \quad (0 \le x \le 1)$$

- Boundary Conditions: Dirichlet

$$u = 0 @ x = 0$$

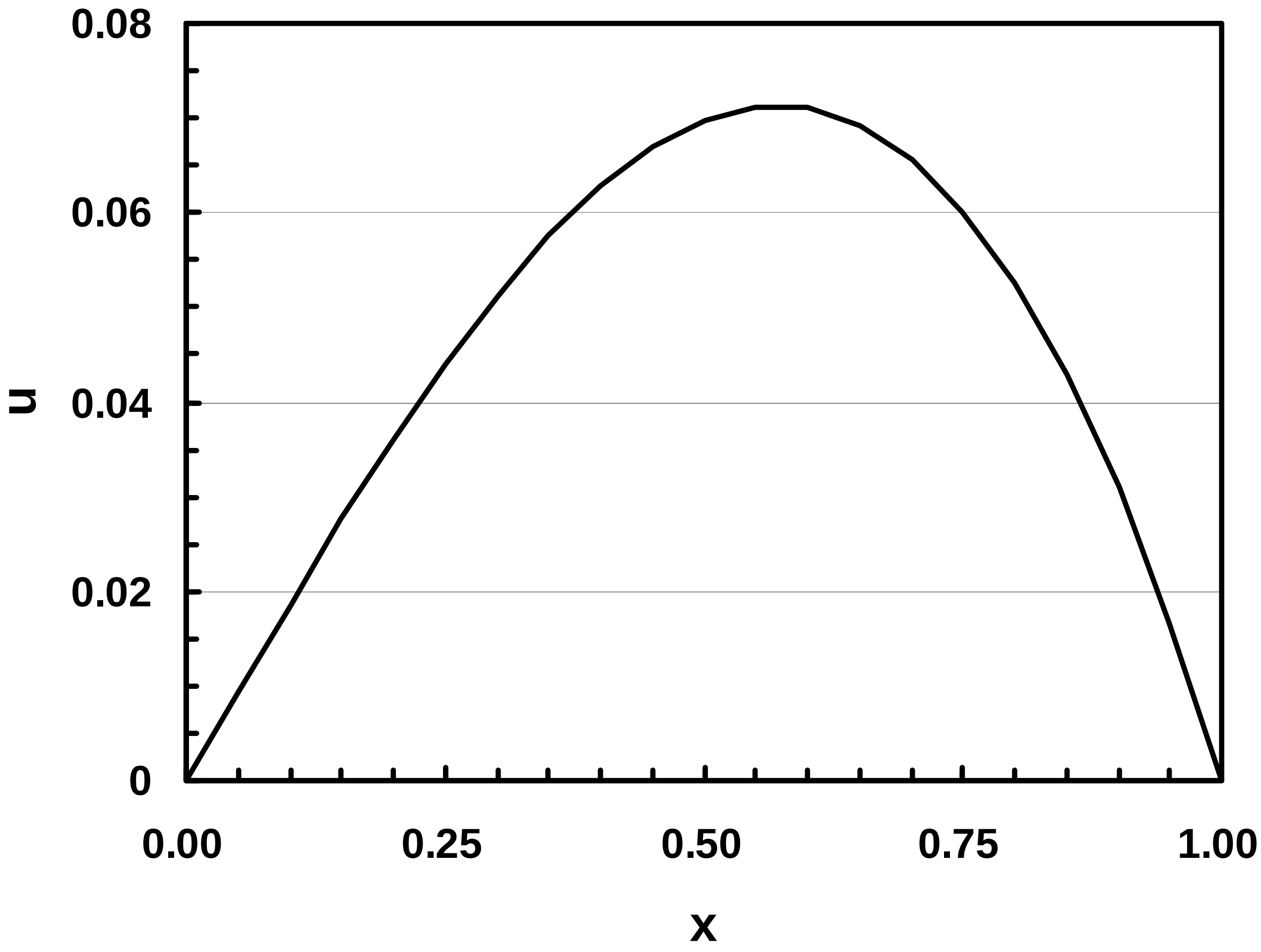
$$u = 0 @ x = 1$$

- Exact Solution

$$u = \frac{\sin x}{\sin 1} - x$$

Exact Solution

$$u = \frac{\sin x}{\sin 1} - x$$



Example (2/2)

- Assume the following approx. solution:

$$u = x(1-x)(a_1 + a_2 x) = x(1-x)a_1 + x^2(1-x)a_2 = a_1\Psi_1 + a_2\Psi_2$$

$$\Psi_1 = x(1-x), \quad \Psi_2 = x^2(1-x)$$

Test/trial function satisfies *u=0@x=0,1*

- Residual is as follows:

$$R(a_1, a_2, x) = x + (-2 + x - x^2)a_1 + (2 - 6x + x^2 - x^3)a_2$$

- Let's apply various types of MWR to this equation
  - We have two unknowns (a_1, a_2), therefore we need two independent weighting functions.

Collocation Method

- *n=2, x=1/4, x=1/2* for collocation points:

$$R(a_1, a_2, \frac{1}{4}) = 0, \quad R(a_1, a_2, \frac{1}{2}) = 0$$

$$R(a_1, a_2, x) = x + (-2 + x - x^2)a_1 + (2 - 6x + x^2 - x^3)a_2$$

- Solution:

$$\begin{bmatrix} 29/16 & -35/64 \\ 7/4 & 7/8 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 1/4 \\ 1/2 \end{Bmatrix} \Rightarrow a_1 = \frac{6}{31}, \quad a_2 = \frac{40}{217}$$

$$u = \frac{x(1-x)}{217}(42 + 40x)$$

Least Square Method

- Weighting functions, Residual:

$$w_1 = \frac{\partial R}{\partial a_1} = -2 + x - x^2, \quad w_2 = \frac{\partial R}{\partial a_2} = 2 - 6x + x^2 - x^3$$

$$R(a_1, a_2, x) = x + (-2 + x - x^2)a_1 + (2 - 6x + x^2 - x^3)a_2$$

- Solution:

$$\int_0^1 R(a_1, a_2, x)\frac{\partial R}{\partial a_1}\,dx = \int_0^1 R(a_1, a_2, x)(-2 + x - x^2)\,dx = 0$$

$$\int_0^1 R(a_1, a_2, x)\frac{\partial R}{\partial a_2}\,dx = \int_0^1 R(a_1, a_2, x)(2 - 6x + x^2 - x^3)\,dx = 0$$

$$\begin{bmatrix} 202 & 101 \\ 707 & 1572 \end{bmatrix}\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 55 \\ 399 \end{Bmatrix} \Rightarrow a_1 = \frac{46161}{246137}, \quad a_2 = \frac{41713}{246137}$$

$$u = \frac{x(1-x)}{246137}(46161 + 41713x)$$

Galerkin Method

- Weighting functions, Residual:

$$w_1 = \Psi_1 = x(1-x), \quad w_2 = \Psi_2 = x^2(1-x)$$

$$R(a_1, a_2, x) = x + (-2 + x - x^2)a_1 + (2 - 6x + x^2 - x^3)a_2$$

- Results:

$$\int_0^1 R(a_1, a_2, x)\,\Psi_1\,dx = \int_0^1 R(a_1, a_2, x)\,(x - x^2)\,dx = 0$$

$$\int_0^1 R(a_1, a_2, x)\,\Psi_2\,dx = \int_0^1 R(a_1, a_2, x)\,(x^2 - x^3)\,dx = 0$$

$$\begin{bmatrix} 3/10 & 3/20 \\ 3/20 & 13/105 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 1/12 \\ 1/20 \end{Bmatrix} \Rightarrow a_1 = \frac{71}{369}, \quad a_2 = \frac{7}{41}$$

$$u = \frac{x(1-x)}{369}(71 + 63x)$$

Results

X	Exact	Collocation	Least Square	Galerkin
0.25	0.04401	0.04493	0.04311	0.04408
0.50	0.06975	0.07143	0.06807	0.06944
0.75	0.06006	0.06221	0.05900	0.06009

- Galerkin Method provides the most accurate solution
  - If functional exists, solutions of variational method and Galerkin method agree.
    - A kind of analytical solution (later of this material)
- Many commercial FEM codes use Galerkin method.
- In this class, Galerkin method is used.
- Least-square may provide robust solution in Navier-Stokes solvers for high Re.

Homework (1/2)

- Apply the following two method is the next page to the same equations:
  - Method of Moment
  - Sub-Domain Method
  - Results at x=0.25, 0.50, 0.75

- Compare the results of “collocation method” on “non-collocaion points” with exact solution
  - Explain the behavior
  - Try different collocation points

Homework (2/2)

- Method of Moment（モーメント法）

$$w_i = \mathbf{x}^{i-1} \quad (i \geq 1)$$

  - Weighting functions ?

- Sub-Domain Method（部分領域法）
  - Domain V is divided into subdomains V_i $(i=1\text{-}n)$, and weighting functions w_i are given as follows:

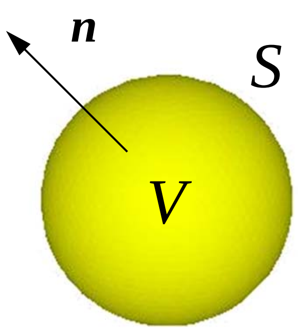
$$w_i = \begin{cases} 1 & \text{for points in } V_i \\ 0 & \text{for points out of } V_i \end{cases}$$

  - Two unknowns, two sub domains

- Numerical Method for PDE (Method of Weighted Residual)
- Gauss-Green’s Theorem
- Numerical Method for PDE (Variational Method)

Gauss's Theorem

$$\int_V \left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} \right) dV = \int_S \left(U n_x + V n_y + W n_z \right) dS$$



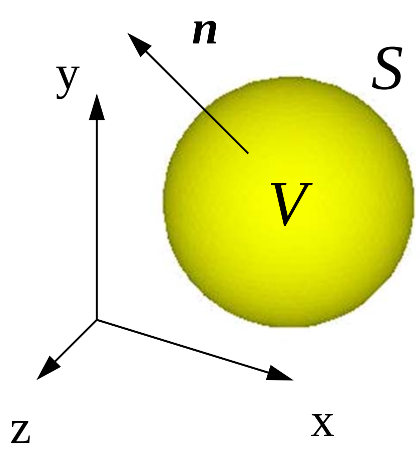
- 3D　(x,y,z)
- Domain V surrounded by smooth closed surface S
- 3 continuous functions defined in V :
  - $U(x,y,z)$,　$V(x,y,z)$,　$W(x,y,z)$
- Outward normal vector $\boldsymbol{n}$ on surface S:
  - n_x,　n_y,　n_z: direction cosine

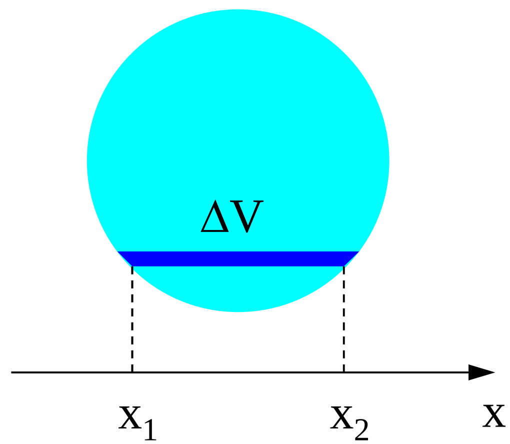
Proof of Gauss's Theorem (1/3)

- Infinitesimal prism which is parallel with x-axis:

$$\int_{\Delta V} \frac{\partial U}{\partial x}\, dV = \iint dy\, dz \int_{x_1}^{x_2} \frac{\partial U}{\partial x}\, dx$$

$$= \iint U(x_2, y, z)\, dy\, dz - \iint U(x_1, y, z)\, dy\, dz$$



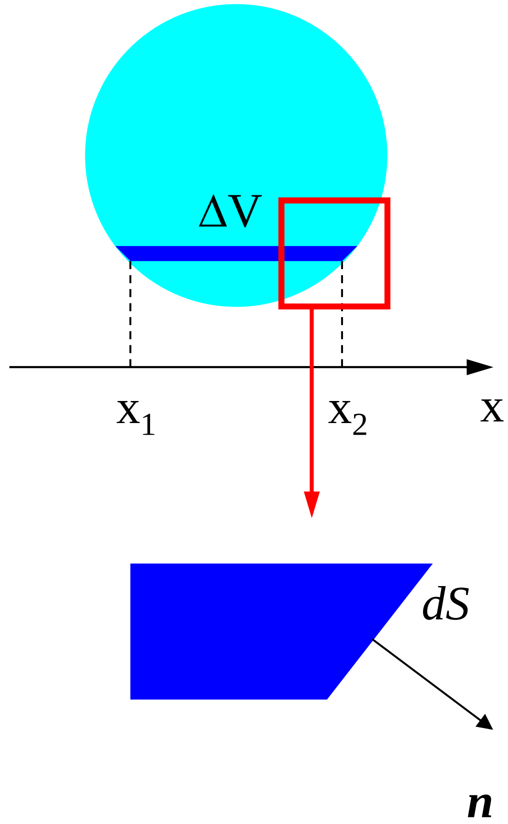


Proof of Gauss's Theorem (2/3)

- Infinitesimal surface *dS* :

$$dy\,dz = +n_x\,dS \quad (if\ \ n_x \geq 0)$$

$$dy\,dz = -n_x\,dS \quad (if\ \ n_x \leq 0)$$



$$\mathbf{n} = \left(n_x, n_y, n_z\right)$$

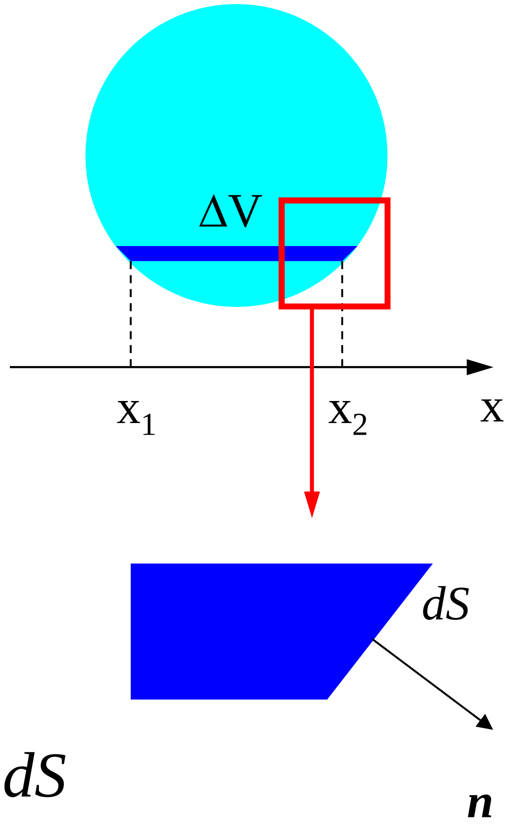
- thus:

$$\int_{\Delta V} \frac{\partial U}{\partial x}\,dV = \iint dy\,dz \int_{x_1}^{x_2} \frac{\partial U}{\partial x}\,dx$$

$$= \iint U(x_2, y, z)\,dy\,dz - \iint U(x_1, y, z)\,dy\,dz$$

$$= \int_{\Delta S_1} U\,n_x\,dS + \int_{\Delta S_2} U\,n_x\,dS$$

Proof of Gauss's Theorem (3/3)



- Integration over the entire surface:

$$\int_V \frac{\partial U}{\partial x}\, dV = \int_S U n_x\, dS$$

- Extension to y-, and z- direction:

$$\int_V \left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} \right) dV = \int_S \left(U n_x + V n_y + W n_z \right) dS$$

$$\mathbf{n} = \left(n_x, n_y, n_z \right)$$

Green's Theorem (1/2)

- Assume the following functions:

$$U = A\frac{\partial B}{\partial x}, \quad V = A\frac{\partial B}{\partial y}, \quad W = A\frac{\partial B}{\partial z}$$

- Thus :

$$\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}+\frac{\partial W}{\partial z} = A\left(\frac{\partial^2 B}{\partial x^2}+\frac{\partial^2 B}{\partial y^2}+\frac{\partial^2 B}{\partial z^2}\right)+\left(\frac{\partial A}{\partial x}\frac{\partial B}{\partial x}+\frac{\partial A}{\partial y}\frac{\partial B}{\partial y}+\frac{\partial A}{\partial z}\frac{\partial B}{\partial z}\right)$$

- Apply Gauss's theorem:

$$\int_V A\left(\frac{\partial^2 B}{\partial x^2}+\frac{\partial^2 B}{\partial y^2}+\frac{\partial^2 B}{\partial z^2}\right)dV + \int_V \left(\frac{\partial A}{\partial x}\frac{\partial B}{\partial x}+\frac{\partial A}{\partial y}\frac{\partial B}{\partial y}+\frac{\partial A}{\partial z}\frac{\partial B}{\partial z}\right)dV$$

$$= \int_S \left(Un_x + Vn_y + Wn_z\right)dS = \int_S A\left(\frac{\partial B}{\partial x}n_x + \frac{\partial B}{\partial y}n_y + \frac{\partial B}{\partial z}n_z\right)dS$$

Green's Theorem (2/2)

- (cont.)

$$\int_S A\left(\frac{\partial B}{\partial x}n_x + \frac{\partial B}{\partial y}n_y + \frac{\partial B}{\partial z}n_z\right)dS = \int_S A\left(\frac{\partial B}{\partial x}\frac{\partial x}{\partial n} + \frac{\partial B}{\partial y}\frac{\partial y}{\partial n} + \frac{\partial B}{\partial z}\frac{\partial z}{\partial n}\right)dS$$

$$= \int_S A\frac{\partial B}{\partial n}dS$$

$\frac{\partial B}{\partial n}$ Gradient of B to the direction of normal vector

- Finally:

$$\int_V A\left(\frac{\partial^2 B}{\partial x^2} + \frac{\partial^2 B}{\partial y^2} + \frac{\partial^2 B}{\partial z^2}\right)dV = \int_S A\frac{\partial B}{\partial n}dS - \int_V \left(\frac{\partial A}{\partial x}\frac{\partial B}{\partial x} + \frac{\partial A}{\partial y}\frac{\partial B}{\partial y} + \frac{\partial A}{\partial z}\frac{\partial B}{\partial z}\right)dV$$

- Appears often after next week
  - From 2nd order differentiation to 1st order differentiation.

In Vector Form

- Gauss's Theorem

$$\int_V \nabla \cdot \mathbf{w} \, dV = \int_S \mathbf{w}^T \mathbf{n} \, dS$$

- Green's Theorem

$$\int_V v \Delta u \, dV = \int_S \left(v \nabla u \right)^T \mathbf{n} \, dS - \int_V \left(\nabla^T v \right) \left(\nabla u \right) dV$$

- Numerical Method for PDE (Method of Weighted Residual)
- Gauss-Green's Theorem
- Numerical Method for PDE (Variational Method)

Variational Method (Ritz) (1/2) 変分法

- It is widely known that exact solution u provides extreme values (max/min) of “functional：汎関数” $I(u)$
  - Euler equation: differential equation satisfied by u, if functional has extreme values（極値）
  - Euler equation is satisfied, if u provides extreme values of $I(u)$.
  - *provide extreme values*：停留させる（or *stationarize*）
- For example, functional, which corresponds to governing equations of linear elasticity (principle of virtual work, equilibrium equations), is “principle of minimum potential energy (principle of minimum strain energy)” .

Variational Method (Ritz) (2/2) 変分法

- Substitute the following approx. solution into $I(u)$, and calculate coefficients a_i under the condition where $I_M=I(u_M)$ provides extreme values, then u_M is obtained:

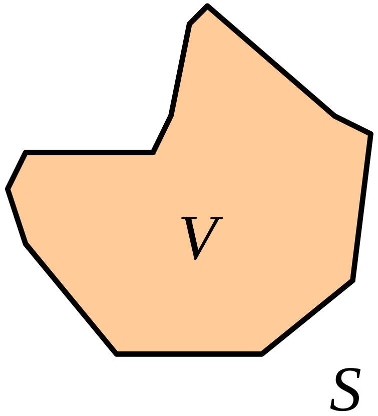
$$u_M = \sum_{i=1}^{M} a_i \Psi_i$$

- Variational method is theoretical method, and can be only applied to differential equations, which has equivalent variational problem.
  - In this class, we mainly use MWR
  - Brief overview of Ritz method will given later today.

Application of Variational Method (1/5)

- Consider the following integration *I(u)* in 2D-domain *V*, where *u(x,y)* is unknown function of *x* and *y*:

$$I(u)=\int_V \frac{1}{2}\left\{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2-2Qu\right\}dV$$



Q: known value

$u=0$ at boundary *S*

- *I(u)* is “functional（汎関数）” of function *u*
- *u** is a twice continuously differentiable function and minimizes *I(u)*. η is an arbitrary function which satisfies η=0 at boundary *S*, and α is a parameter. Consider the following equation:

$$u(x,y)=u^*(x,y)+\alpha\cdot\eta(x,y)$$

Application of Variational Method (2/5)

- At this stage, the following condition is necessary:

  $$I(u) \geq I(u^*)$$

- Assume that functional $I(u^* + \alpha\eta)$ is a function of α. Functional I provides minimum value, if $\alpha=0$. Therefore, the following equation is obtained:

  $$\frac{\partial}{\partial \alpha} I(u^* + \alpha \cdot \eta)\Big|_{\alpha=0} = 0$$

- According to the definition of functional $I(u)$, following equation is obtained

  $$\int_V \left(\frac{\partial u^*}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial u^*}{\partial y}\frac{\partial \eta}{\partial y} - Q\eta \right) dV = 0$$

$$I(u) = \int_V \frac{1}{2}\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 - 2Qu\right\} dV$$

$$u(x,y) = u^*(x,y) + \alpha \cdot \eta(x,y)$$

$$\frac{\partial}{\partial \alpha} I\left(u^* + \alpha \cdot \eta\right)\Big|_{\alpha=0} = 0$$

$$\frac{\partial}{\partial \alpha}\left\{\frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2\right\} = \frac{\partial u}{\partial x} \cdot \frac{\partial}{\partial \alpha}\left(\frac{\partial u}{\partial x}\right), \quad \frac{\partial u}{\partial x} = \frac{\partial\left(u^* + \alpha \cdot \eta\right)}{\partial x} = \frac{\partial u^*}{\partial x} + \alpha \frac{\partial \eta}{\partial x}$$

$$\frac{\partial}{\partial \alpha}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial \eta}{\partial x}, \quad \alpha = 0 \Rightarrow \frac{\partial}{\partial \alpha}\left\{\frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2\right\} = \frac{\partial u^*}{\partial x}\frac{\partial \eta}{\partial x}, \quad \frac{\partial}{\partial \alpha}\left\{\frac{1}{2}\left(\frac{\partial u}{\partial y}\right)^2\right\} = \frac{\partial u^*}{\partial y}\frac{\partial \eta}{\partial y}$$

$$\frac{\partial}{\partial \alpha}(Qu) = Q\frac{\partial\left(u^* + \alpha \cdot \eta\right)}{\partial \alpha} = Q\eta$$

$$\int_V \left(\frac{\partial u^*}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial u^*}{\partial y}\frac{\partial \eta}{\partial y} - Q\eta\right) dV = 0$$

Application of Variational Method (3/5)

- Apply Green's theorem on 1st and 2nd term of LHS, and apply integration by parts, then following equation is obtained: ($A=\eta$, $B=u*$) :

$$-\int_V \left(\frac{\partial^2 u^*}{\partial x^2} + \frac{\partial^2 u^*}{\partial y^2} + Q \right) \eta \, dV + \int_S \eta \frac{\partial u^*}{\partial n} \, dS = 0$$

$$where \quad \frac{\partial u^*}{\partial n} = \frac{\partial u^*}{\partial x} n_x + \frac{\partial u^*}{\partial y} n_y$$

Gradient of $u*$ in the direction of normal vector

- At boundary S, η=0:

$$-\int_V \left(\frac{\partial^2 u^*}{\partial x^2} + \frac{\partial^2 u^*}{\partial y^2} + Q \right) \eta \, dV = 0$$

- (A) is required, if the above is true for arbitrary η

$$\frac{\partial^2 u^*}{\partial x^2} + \frac{\partial^2 u^*}{\partial y^2} + Q = 0 \quad \text{(A)}$$

Green's Theorem

- $(A=\eta,\ B=u^*)$:

$$\int_V \left(\frac{\partial u^*}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial u^*}{\partial y}\frac{\partial \eta}{\partial y} - Q\eta \right) dV = 0$$

$$\int_V \eta \left(\frac{\partial^2 u^*}{\partial x^2} + \frac{\partial^2 u^*}{\partial y^2} \right) dV = \int_S \eta \frac{\partial u^*}{\partial n}\, dS - \int_V \left(\frac{\partial \eta}{\partial x}\frac{\partial u^*}{\partial x} + \frac{\partial \eta}{\partial y}\frac{\partial u^*}{\partial y} \right) dV$$

$$\int_V \left(\frac{\partial \eta}{\partial x}\frac{\partial u^*}{\partial x} + \frac{\partial \eta}{\partial y}\frac{\partial u^*}{\partial y} \right) dV \int_V = -\eta \left(\frac{\partial^2 u^*}{\partial x^2} + \frac{\partial^2 u^*}{\partial y^2} \right) dV + \int_S \eta \frac{\partial u^*}{\partial n}\, dS$$

Application of Variational Method (4/5)

- Equation (A) is called “Euler equation”
  - Necessary condition of $u*$, which minimizes functional $I(u)$, is that $u*$ satisfies the Euler equation.
- Sufficient condition:
  - Assume that u^* is solution of the Euler equation and $\alpha\eta=\delta u^*$

$$I\left(u^* + \delta u^*\right) - I\left(u^*\right) =$$

$$-\int_V \left(\frac{\partial^2 u^*}{\partial x^2} + \frac{\partial^2 u^*}{\partial y^2} + Q \right) \delta u^* \, dV + \int_V \frac{1}{2} \left\{ \left(\frac{\partial \left(\delta u^*\right)}{\partial x} \right)^2 + \left(\frac{\partial \left(\delta u^*\right)}{\partial y} \right)^2 \right\} dV$$

$\delta I = 0$
First Variation
第一変分

$\delta I^2 \geqq 0$
Second Variation
第二変分

Application of Variational Method (5/5)

- It has been proved that u^* (solution of Euler equation) minimizes functional $I(u)$.

$$I\left(u^* + \delta u^*\right) \geq I\left(u^*\right)$$

- Therefore, boundary value problem by Euler equation (A) with B.C. (u=0) is equivalent to variational problem.
  - Solving equivalent variational problem provides solution of Euler equation (Poission equation in this case)
  - Functional must exist !

Approx. by Variational Method (1/4)

- Functional

$$I(u)=\int_0^1 \left\{\frac{1}{2}\left(\frac{du}{dx}\right)^2-\frac{1}{2}u^2-xu\right\}dx$$

- Boundary Condition

$$u=0 @ x=0$$

$$u=0 @ x=1$$

- Obtain u, which “stationalizes” functional $I(u)$ under this B.C.
  - Corresponding Euler equation is as follows (same as equation in p.21):

$$\frac{d^2u}{dx^2}+u+x=0 \quad (0\le x\le 1) \qquad \text{(B-1)}$$

Approx. by Variational Method (2/4)

- Assume the following test function with n-th order for function u, which is twice continuously differentiable:

$$u_n = x \cdot (1 - x) \cdot \left(a_1 + a_2 x + a_3 x^2 + \cdots + a_n x^{n-1} \right) \quad \text{(B-2)}$$

- If we increase the order of test function, u_n is closer to exact solution u. Therefore, functional $I(u)$ can be approximated by $I(u_n)$:
  - If $I(u_n)$ stationarizes, $I(u)$ also stationarizes.
- We need to obtain set of unknown coefficients a_k, which satisfies the following stationary condition:

$$\frac{\partial I(u_n)}{\partial a_k} = 0 \quad (k = 1 \sim n) \quad \text{(B-3)}$$

Ritz Method

- Equation (B-3) is linear equations for a_1-a_n.
- If this solutions is applied to equation (B-2), approximate solution, which satisfies Euler equation (B-1), is obtained.
  - Approximate solution, but satisfies Euler equation strictly （厳密解）
- This type of method using a set of coefficients a_1-a_n is called “Ritz Method”.

Approx. by Variational Method (3/4)

- Ritz Method, n=2

$$u_2 = x\cdot(1-x)\cdot(a_1 + a_2 x) = x\cdot(1-x)\cdot a_1 + x^2\cdot(1-x)\cdot a_2$$

$$\frac{\partial I(u_2)}{\partial a_1} = 0 \Rightarrow \left[\int_0^1 (1-x-x^2)(1-3x+x^2)dx\right]a_1$$

$$+\left[\int_0^1 \left\{(1-2x)(2x-3x^2) - x^3(1-x)^2\right\}dx\right]a_2 + \int_0^1 x^2(1-x)dx = 0$$

$$\frac{\partial I(u_2)}{\partial a_2} = 0 \Rightarrow \left[\int_0^1 \left\{(1-2x)(2x-3x^2) - x^3(1-x)^2\right\}dx\right]a_1$$

$$+\left[\int_0^1 (2x-3x^2+x^3)(2x-2x^2-x^3)dx\right]a_2 + \int_0^1 x^3(1-x)dx = 0$$

Supplementation for (3/4) (1/3)

- Ritz Method, *n*=2

$$u_2 = x\cdot(1-x)\cdot(a_1+a_2x) = x\cdot(1-x)\cdot a_1 + x^2\cdot(1-x)\cdot a_2$$

$$I(u) = \int_0^1 \left\{ \frac{1}{2}\left(\frac{du}{dx}\right)^2 - \frac{1}{2}u^2 - xu \right\} dx$$

$$\frac{1}{2}\left(\frac{du}{dx}\right)^2 - \frac{1}{2}u^2 - xu =$$

$$\frac{1}{2}\left[(1-2x)a_1 + \left(2x-3x^2\right)a_2\right]^2 - \frac{1}{2}\left[x\cdot(1-x)\cdot a_1 + x^2\cdot(1-x)\cdot a_2\right]^2$$
$$-\left[x^2\cdot(1-x)\cdot a_1 + x^3\cdot(1-x)\cdot a_2\right]$$

Supplementation for (3/4) (2/3)

$$\frac{1}{2}\left(\frac{du}{dx}\right)^2-\frac{1}{2}u^2-xu=$$

$$\frac{1}{2}\left[(1-2x)a_1+\left(2x-3x^2\right)a_2\right]^2-\frac{1}{2}\left[x\cdot(1-x)\cdot a_1+x^2\cdot(1-x)\cdot a_2\right]^2$$
$$-\left[x^2\cdot(1-x)\cdot a_1+x^3\cdot(1-x)\cdot a_2\right]$$

$$\frac{\partial I(u_2)}{\partial a_1}=0\Rightarrow$$

$$\left[\int_0^1\left\{(1-2x)^2-x^2\cdot(1-x)^2\right\}dx\right]a_1$$
$$+\left[\int_0^1\left\{(1-2x)\left(2x-3x^2\right)-x^3\cdot(1-x)^2\right\}dx\right]a_2-\int_0^1 x^2\cdot(1-x)dx=0$$

Supplementation for (3/4) (3/3)

$$\frac{1}{2}\left(\frac{du}{dx}\right)^2 - \frac{1}{2}u^2 - xu =$$

$$\frac{1}{2}\left[(1-2x)a_1 + \left(2x-3x^2\right)a_2\right]^2 - \frac{1}{2}\left[x\cdot(1-x)\cdot a_1 + x^2\cdot(1-x)\cdot a_2\right]^2$$

$$-\left[x^2\cdot(1-x)\cdot a_1 + x^3\cdot(1-x)\cdot a_2\right]$$

$$\frac{\partial I(u_2)}{\partial a_2} = 0 \Rightarrow$$

$$\left[\int_0^1 \left\{(1-2x)\left(2x-3x^2\right) - x^3\cdot(1-x)^2\right\}dx\right]a_1$$

$$+\left[\int_0^1 \left\{\left(2-3x^2\right)^2 - x^4\cdot(1-x)^2\right\}dx\right]a_2 - \int_0^1 x^3\cdot(1-x)dx = 0$$

Approx. by Variational Method (4/4)

- Final linear equations are as follows:

$$\begin{bmatrix} 3/10 & 3/20 \\ 3/20 & 13/105 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 1/12 \\ 1/20 \end{Bmatrix} \Rightarrow a_1 = \frac{71}{369}, \quad a_2 = \frac{7}{41}$$

$$u = \frac{x(1-x)}{369}(71 + 63x)$$

- This result is identical with that of Galerkin Method
  - NOT a coincidence !!

Galerkin Method

- Weighting functions (which satisfy u=0@x=0,1), Residual:

$$w_1 = \Psi_1 = x(1-x), \quad w_2 = \Psi_2 = x^2(1-x)$$

$$R(a_1, a_2, x) = x + (-2 + x - x^2)a_1 + (2 - 6x + x^2 - x^3)a_2$$

- Results:

$$\int_0^1 R(a_1, a_2, x)\,\Psi_1\,dx = \int_0^1 R(a_1, a_2, x)\,(x - x^2)\,dx = 0$$

$$\int_0^1 R(a_1, a_2, x)\,\Psi_2\,dx = \int_0^1 R(a_1, a_2, x)\,(x^2 - x^3)\,dx = 0$$

$$\begin{bmatrix} 3/10 & 3/20 \\ 3/20 & 13/105 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 1/12 \\ 1/20 \end{Bmatrix} \Rightarrow a_1 = \frac{71}{369}, \quad a_2 = \frac{7}{41}$$

$$u = \frac{x(1-x)}{369}(71 + 63x)$$

Ritz Method & Galerkin Method (1/4)

$$u_2 = x \cdot (1 - x) \cdot (a_1 + a_2 x) = a_1 w_1 + a_2 w_2$$

$$I(u) = \int_0^1 \left\{ \frac{1}{2} \left(\frac{du}{dx} \right)^2 - \frac{1}{2} u^2 - xu \right\} dx$$

$$\frac{\partial}{\partial a_1} \left[\frac{1}{2} \left(\frac{du_2}{dx} \right)^2 \right] = \frac{du_2}{dx} \cdot \frac{\partial}{\partial a_1} \left(\frac{du_2}{dx} \right) = \left(a_1 \frac{dw_1}{dx} + a_2 \frac{dw_2}{dx} \right) \frac{dw_1}{dx}$$

$$\frac{\partial}{\partial a_1} \left[\frac{1}{2} u_2^2 \right] = u_2 \cdot \frac{\partial u_2}{\partial a_1} = (a_1 w_1 + a_2 w_2) \cdot w_1$$

$$\frac{\partial}{\partial a_1} [x u_2] = x \cdot \frac{\partial u_2}{\partial a_1} = x \cdot w_1$$

$$\frac{\partial I(u_2)}{\partial a_1} = 0 \Rightarrow$$

$$\left[\int_0^1 \left\{ \left(\frac{dw_1}{dx} \right)^2 a_1 + \frac{dw_1}{dx} \frac{dw_2}{dx} a_2 \right\} dx \right] - \left[\int_0^1 w_1 \{ (w_1 a_1 + w_2 a_2) + x \} dx \right] = 0$$

$$\frac{\partial I(u_2)}{\partial a_2} = 0 \Rightarrow$$

$$\left[\int_0^1 \left\{ \frac{dw_1}{dx} \frac{dw_2}{dx} a_1 + \left(\frac{dw_2}{dx} \right)^2 a_2 \right\} dx \right] - \left[\int_0^1 w_2 \{ (w_1 a_1 + w_2 a_2) + x \} dx \right] = 0$$

Ritz Method & Galerkin Method (2/4)

$$\frac{\partial I(u_2)}{\partial a_1} = 0 \Rightarrow$$

$$\left[\int_0^1 \left\{ \left(\frac{dw_1}{dx}\right)^2 a_1 + \frac{dw_1}{dx}\frac{dw_2}{dx} a_2 \right\} dx \right] - \left[\int_0^1 w_1 \{(w_1 a_1 + w_2 a_2) + x\} dx \right] = 0$$

$$w_1 = \Psi_1 = x(1-x),$$

$$w_2 = \Psi_2 = x^2(1-x)$$

$$\frac{\partial}{\partial x}\left(w_1 \frac{dw_1}{dx}\right) = \frac{dw_1}{dx}\frac{dw_1}{dx} + w_1 \frac{d^2 w_1}{dx^2}$$

$$\frac{\partial}{\partial x}\left(w_1 \frac{dw_2}{dx}\right) = \frac{dw_1}{dx}\frac{dw_2}{dx} + w_1 \frac{d^2 w_2}{dx^2}$$

$$\int_0^1 \left\{ \left(\frac{dw_1}{dx}\right)^2 a_1 \right\} dx = \left(a_1 w_1 \frac{dw_1}{dx} \right)\Bigg|_0^1 - \int_0^1 w_1 \left\{ \frac{d^2 w_1}{dx^2} a_1 \right\} dx = -\int_0^1 w_1 \left\{ \frac{d^2 w_1}{dx^2} a_1 \right\} dx$$

$$\int_0^1 \left\{ \left(\frac{dw_1}{dx}\frac{dw_2}{dx}\right) a_2 \right\} dx = \left(a_2 w_1 \frac{dw_2}{dx} \right)\Bigg|_0^1 - \int_0^1 w_1 \left\{ \frac{d^2 w_2}{dx^2} a_2 \right\} dx = -\int_0^1 w_1 \left\{ \frac{d^2 w_2}{dx^2} a_2 \right\} dx$$

Ritz Method & Galerkin Method (3/4)

$$\frac{d^2u}{dx^2}+u+x=0$$

$$u=a_1w_1+a_2w_2$$

$$\frac{\partial I(u_2)}{\partial a_1}=0\Rightarrow$$

$$-\int_0^1 w_1\left\{\left(\frac{d^2w_1}{dx^2}a_1+\frac{d^2w_2}{dx^2}a_2\right)+\left(w_1a_1+w_2a_2\right)+x\right\}dx=0$$

$$-\int_0^1 w_1\left(\frac{d^2u_2}{dx^2}+u_2+x\right)dx=0$$

Galerkin Method !!

$$\frac{\partial I(u_2)}{\partial a_2}=0\Rightarrow$$

$$-\int_0^1 w_2\left\{\left(\frac{d^2w_1}{dx^2}a_1+\frac{d^2w_2}{dx^2}a_2\right)+\left(w_1a_1+w_2a_2\right)+x\right\}dx=0$$

$$-\int_0^1 w_2\left(\frac{d^2u_2}{dx^2}+u_2+x\right)dx=0$$

Ritz Method & Galerkin Method (4/4)

- This example is a very special case. But, generally speaking, results of Galerkin method and Ritz method agree, if functional exists.
- Although Ritz method provides approx. solution, that satisfies Euler equation in strict sense. Therefore, solution of Ritz method is closer to exact solution.
  - This is the main reason that Galerkin method is accurate.
    - Please just remember this.
- This relationship between Ritz and Galerkin is not correct if functional does not exist.
  - In these cases, Galerkin method is not necessarily the best method from the viewpoint of accuracy and robustness.