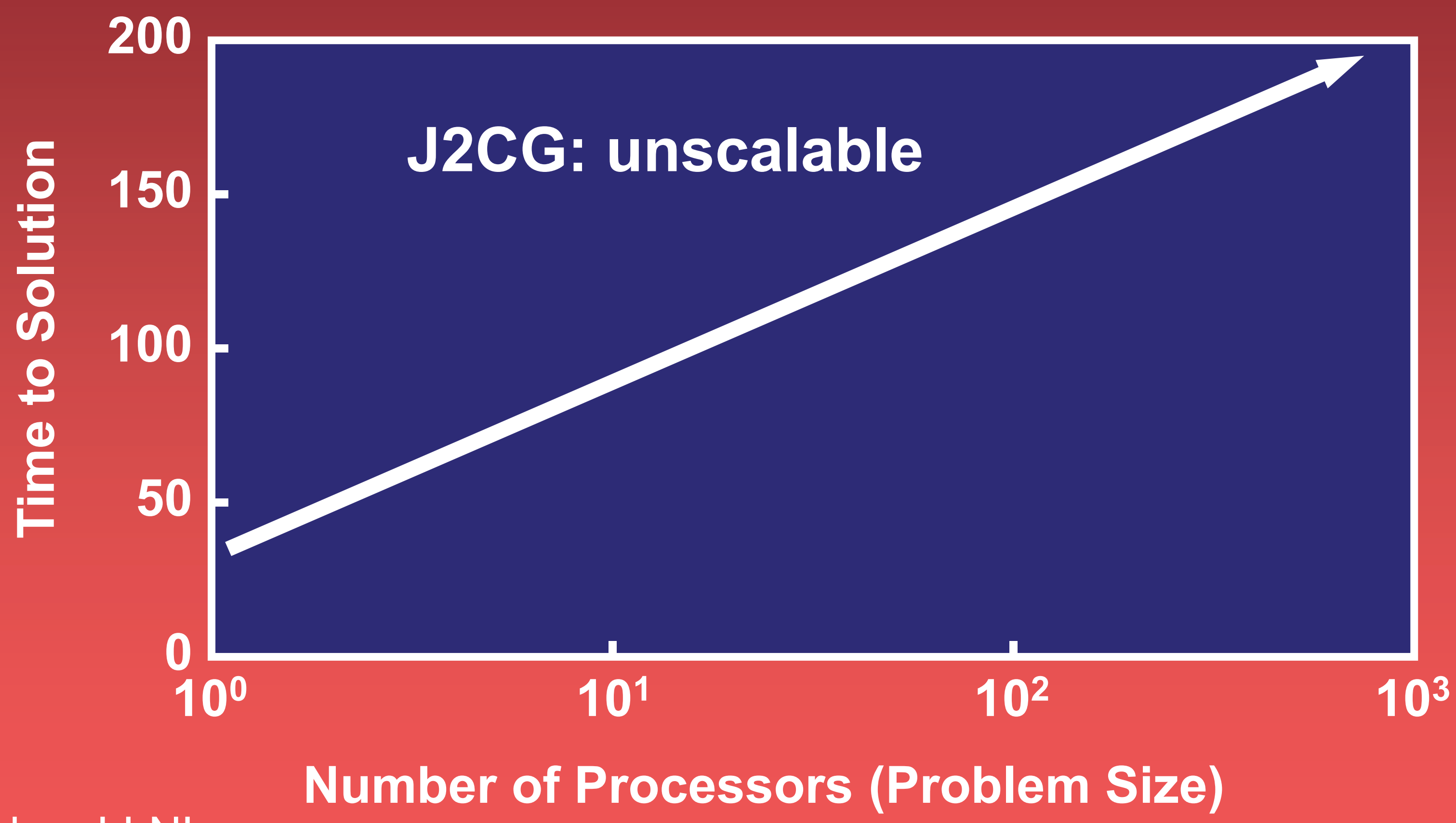
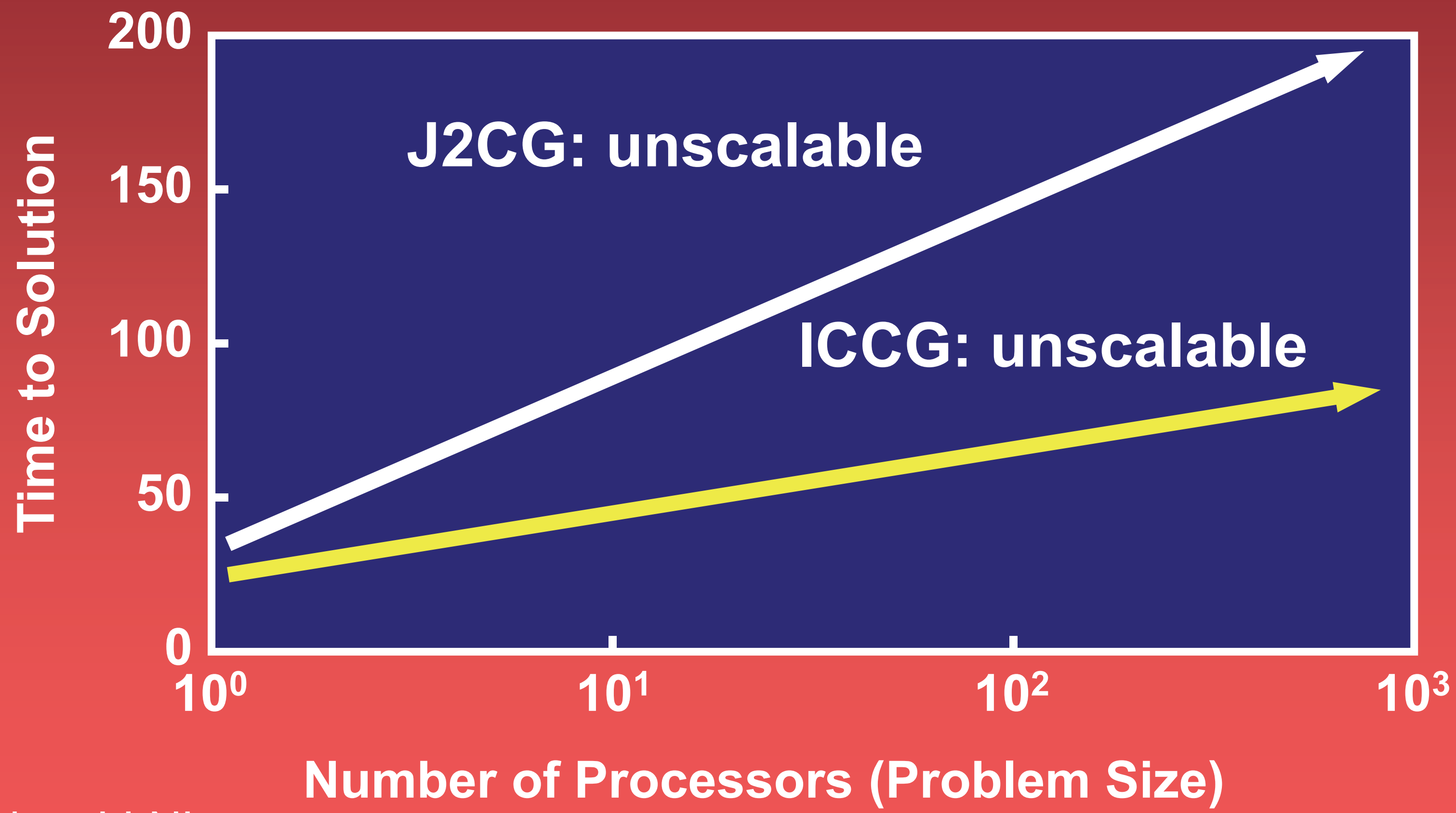


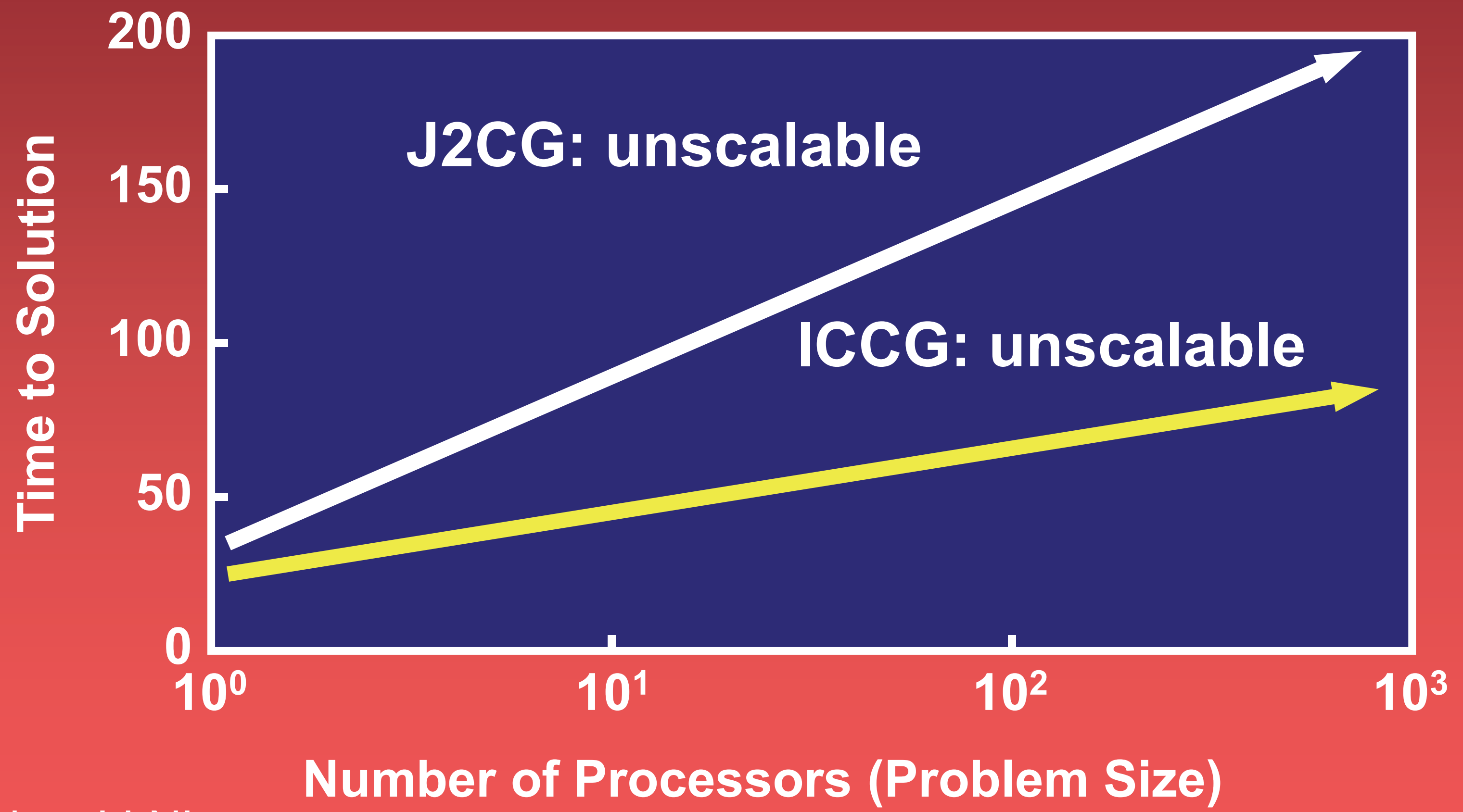
Based on LLNL

ICCG is good …



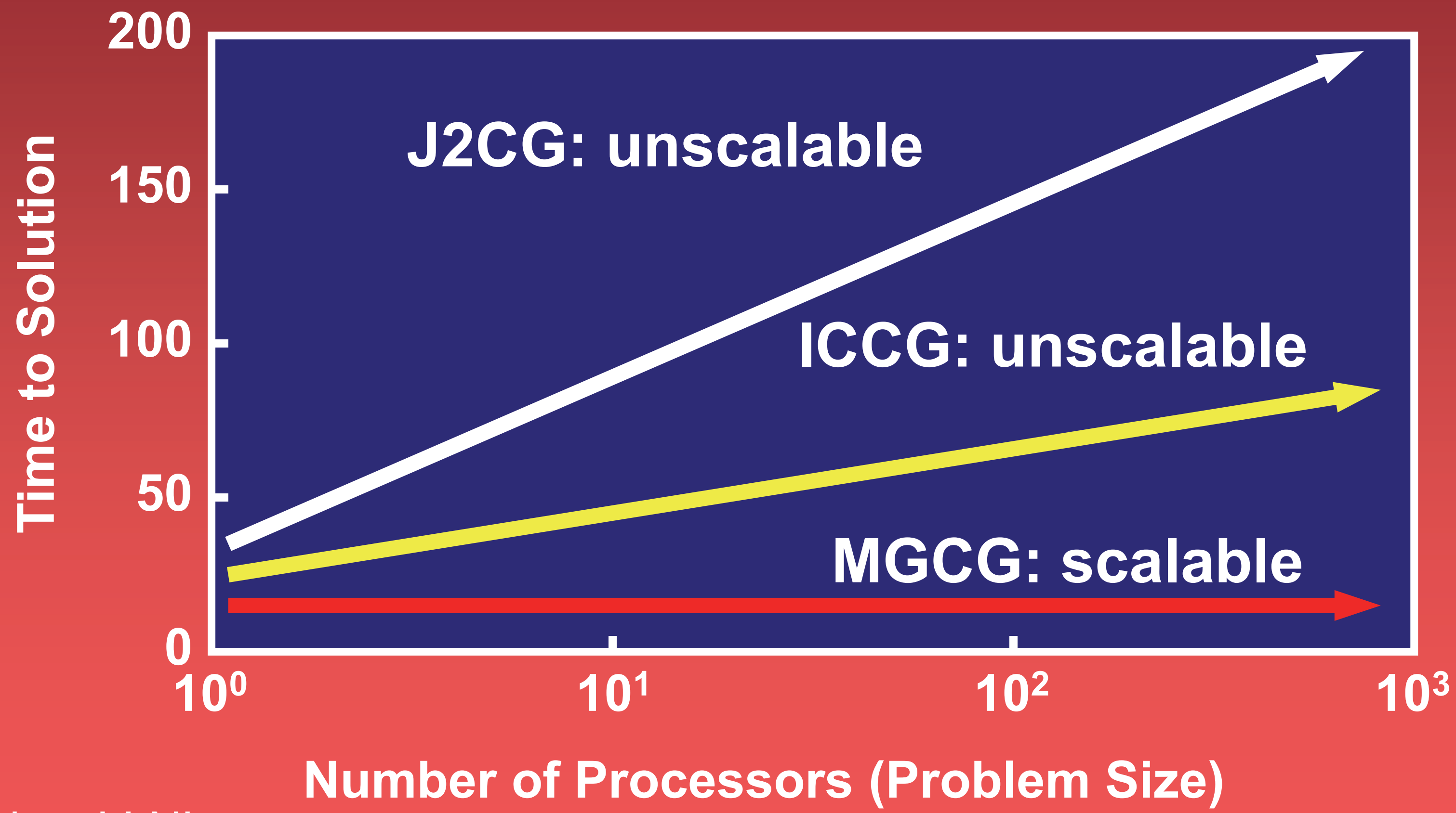
Based on LLNL

ICCG is good **but not scalable**



Based on LLNL

MGCG is scalable

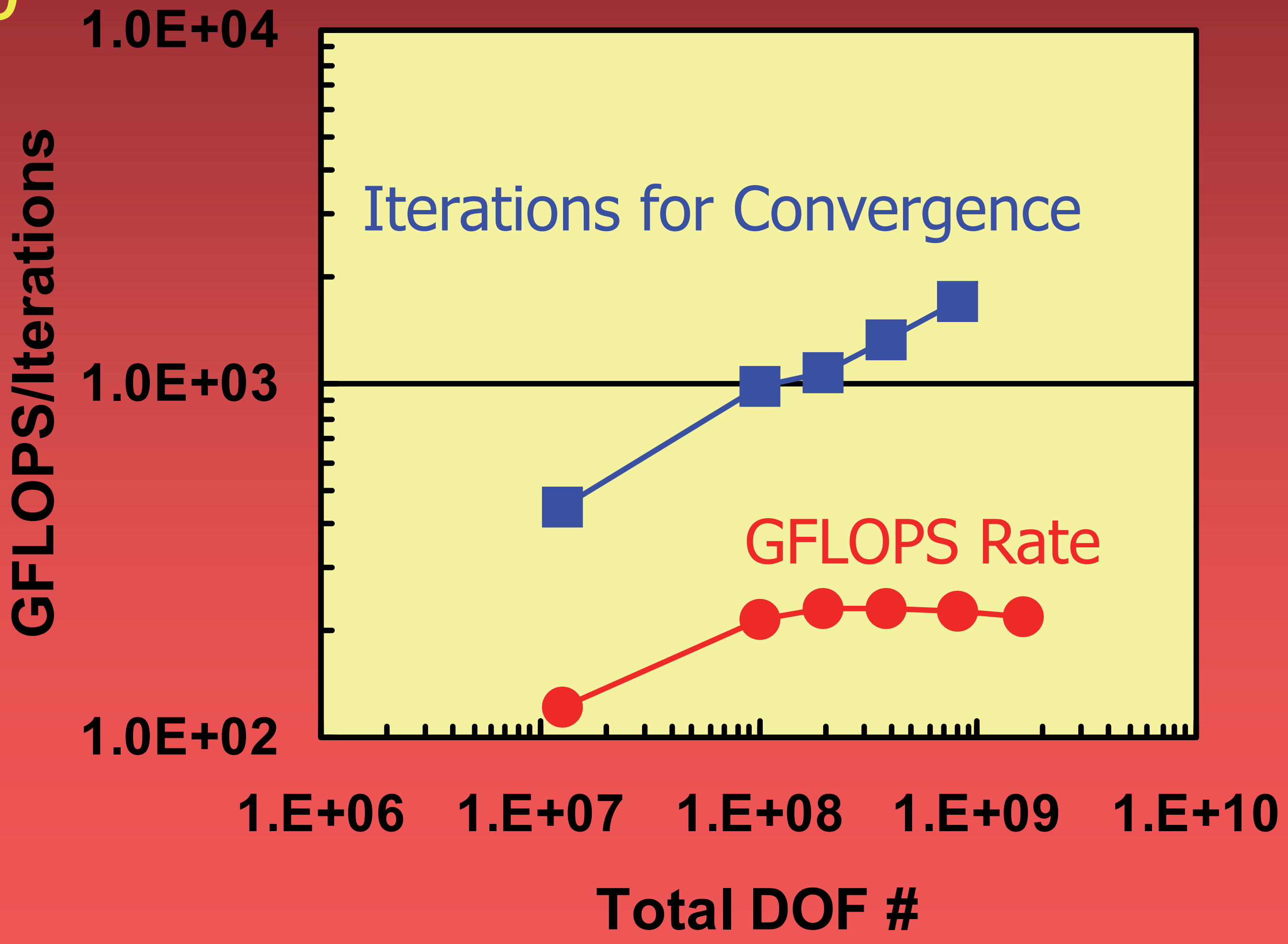


Based on LLNL

Experience in GeoFEM

FLOPS rate is not everything !!

Results on 128 nodes (1024 PEs) of Hitachi SR8000



Outline of this Study

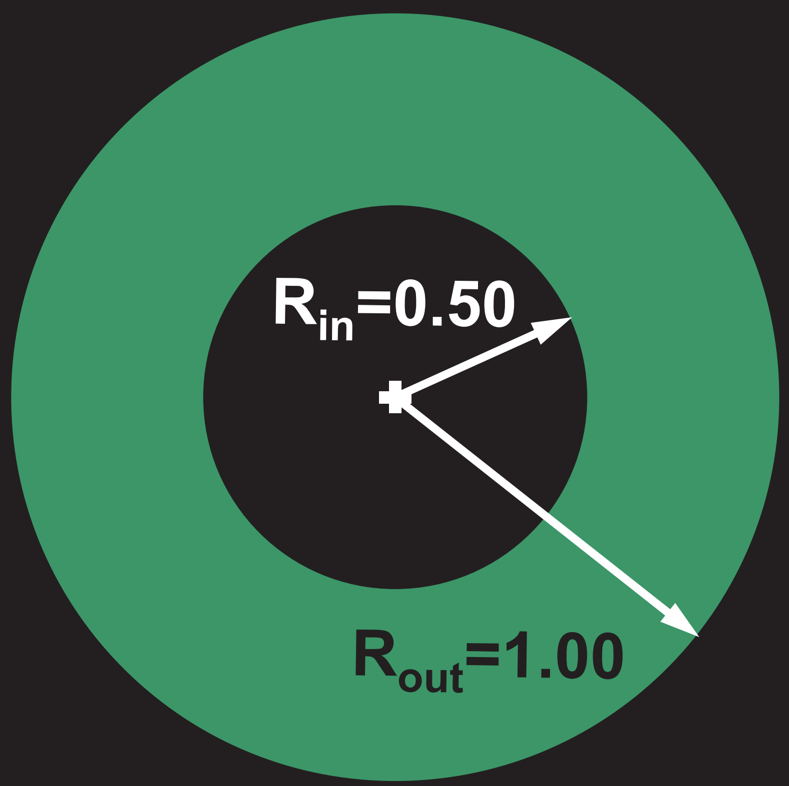
- Parallel Multigrid Method for Region between 2 Spherical Surfaces
- Geometric Multigird Procedure
  - Semi-Unstructured Prisms
  - Adaptively Refined Hierarchical Mesh : starting from Icosahedron
  - Effect of Local Refinement

Overview

- Target Application
  - Governing Equations, Method
  - Geometry
  - Mesh Generation by Adaptive Mesh Refinement (AMR)
- Parallel Multigrid Procedure
  - V-Cycle Multigrid
  - Data Structure
- Examples
  - Large-Scale Examples
  - Effect of Local Refinement
- Future Study

Target Application

Thermal Convection between 2 Spherical Surfaces



- Typical Geometry in Earth Science Simulations
  - Mantle, Core etc.
  - Also applicable to external flow
- Boundary Conditions
  - $r = R_{in}$
    - $u=v=w=0$, $T=1$
  - $r = R_{out}$
    - $u=v=w=0$, $T=0$
  - Heat Generation

Governing Equations

3D Laminar Viscous Incompressible Flow with Natural Convection

- Momentum Equations (Navier-Stokes)

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - \frac{1}{\mathrm{Re}}\Delta\mathbf{u} + \mathbf{g}\frac{Gr}{\mathrm{Re}^2}\Delta T = 0$$

- Continuity Equation

$$\nabla\cdot\mathbf{u} = 0$$

- Energy Equation

$$\frac{\partial \mathrm{T}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathrm{T} - \frac{1}{\mathrm{Re}\,\mathrm{Pr}}\Delta\mathrm{T} = \mathrm{Q}$$

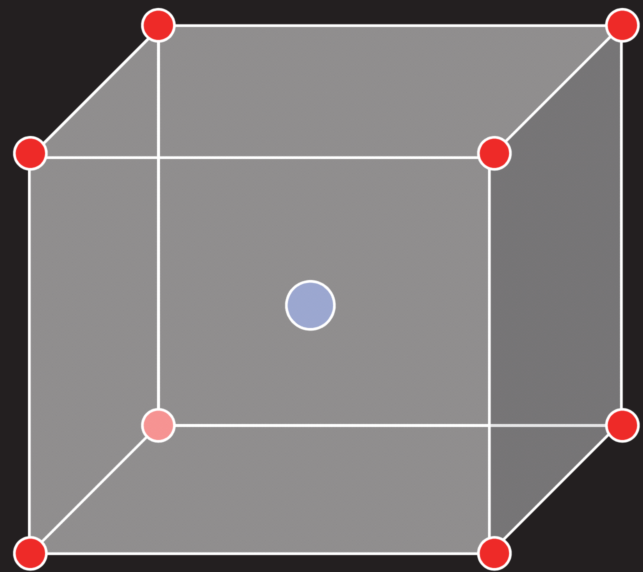
Semi-Implicit Pressure Correction Scheme

- Solve momentum equations explicitly
- Solve pressure correction Poisson equation and update velocity field so that continuity would be satisfied.
- Solve energy equation explicitly
- Poisson equation solver
  - CG
  - Most time consuming part
  - Multilevel implementation in this work

Staggered Grids

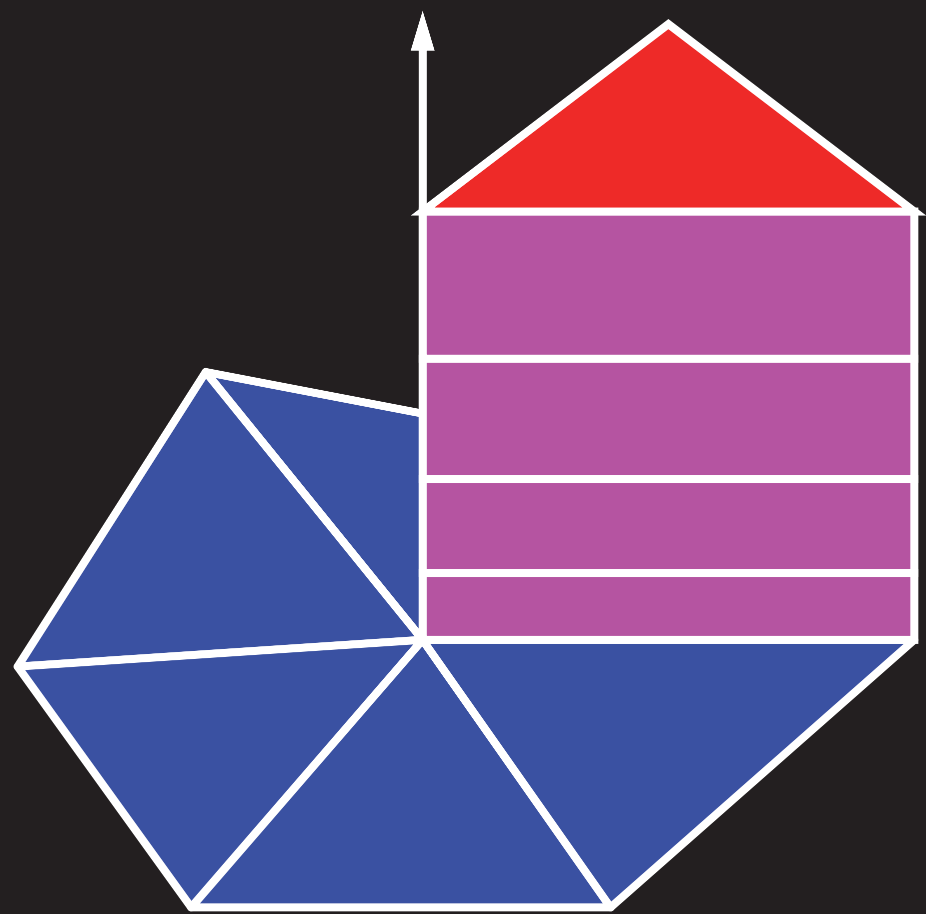
- Gauss-Green Type Finite Volume Method

- Corner Nodes
  - Velocity Components
  - Temperature
- Cell Center
  - Pressure
  - Pressure Correction Potential
  - Material Property



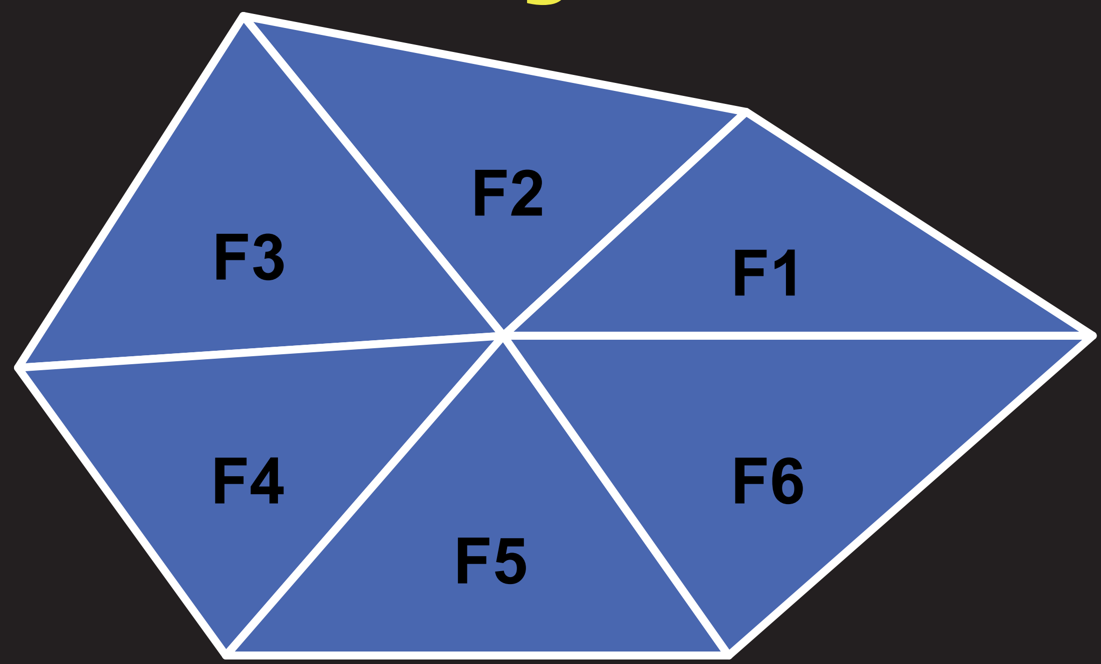
Semi-Unstructured Prismatic Grids

- generated from unstructured surface triangles
- structured in normal-to-surface direction
- flexible
- suitable for near-wall boundary layer computation



Semi-Unstructured Prismatic Grids

Generated from Surface Triangles



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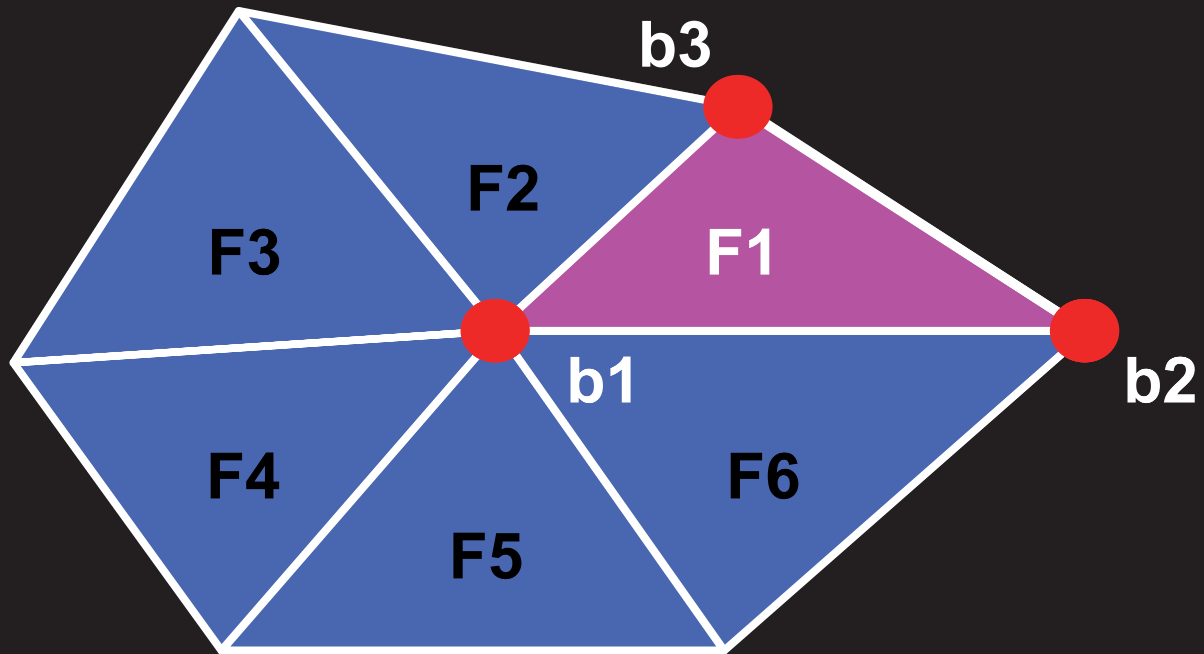

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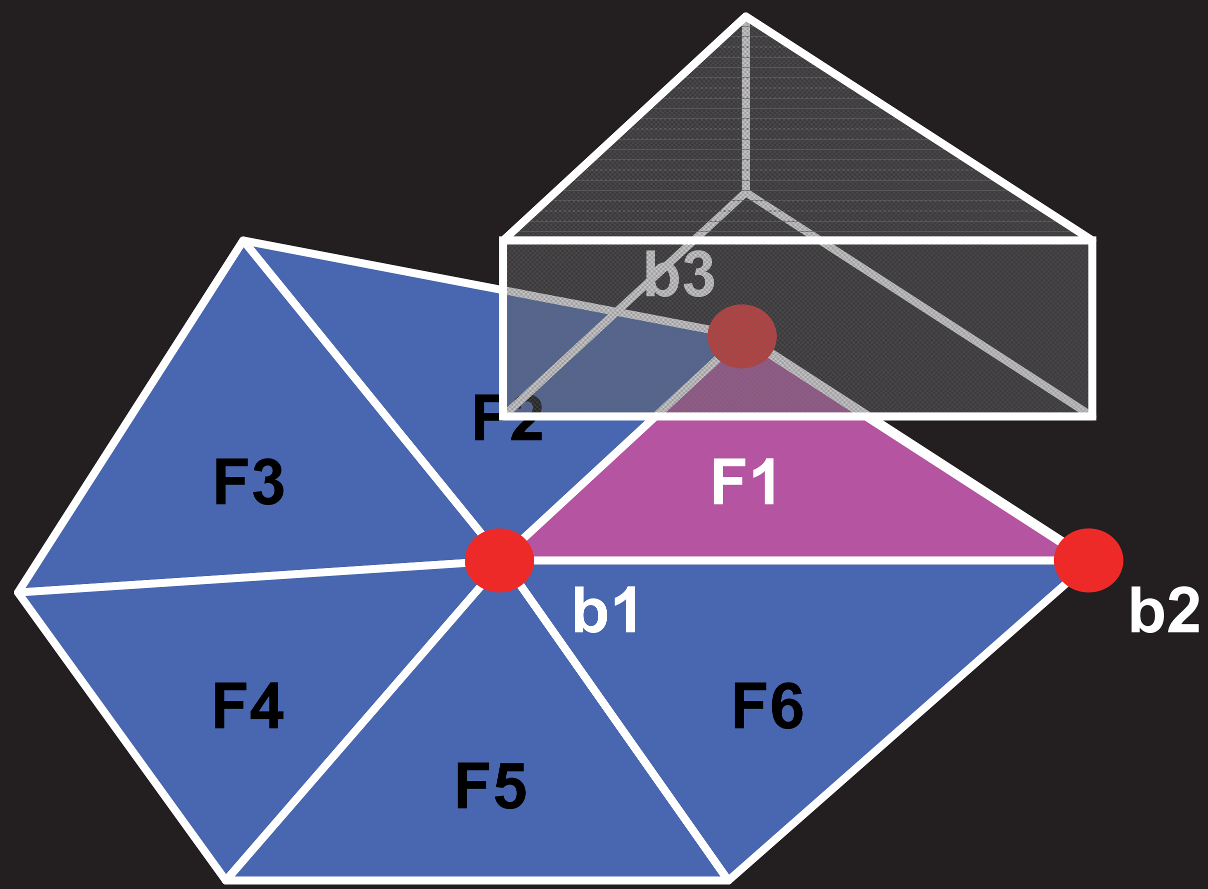
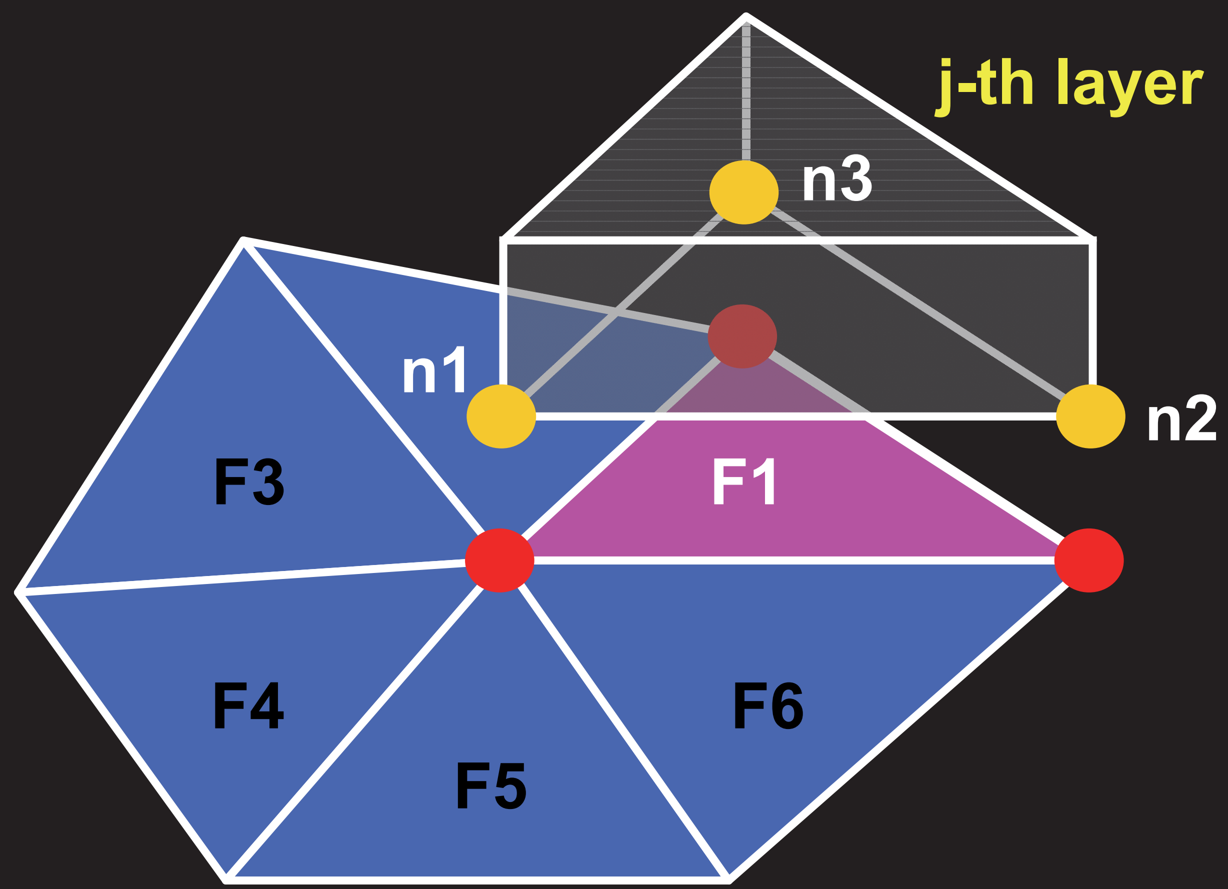


Semi-Unstructured Prismatic Grids



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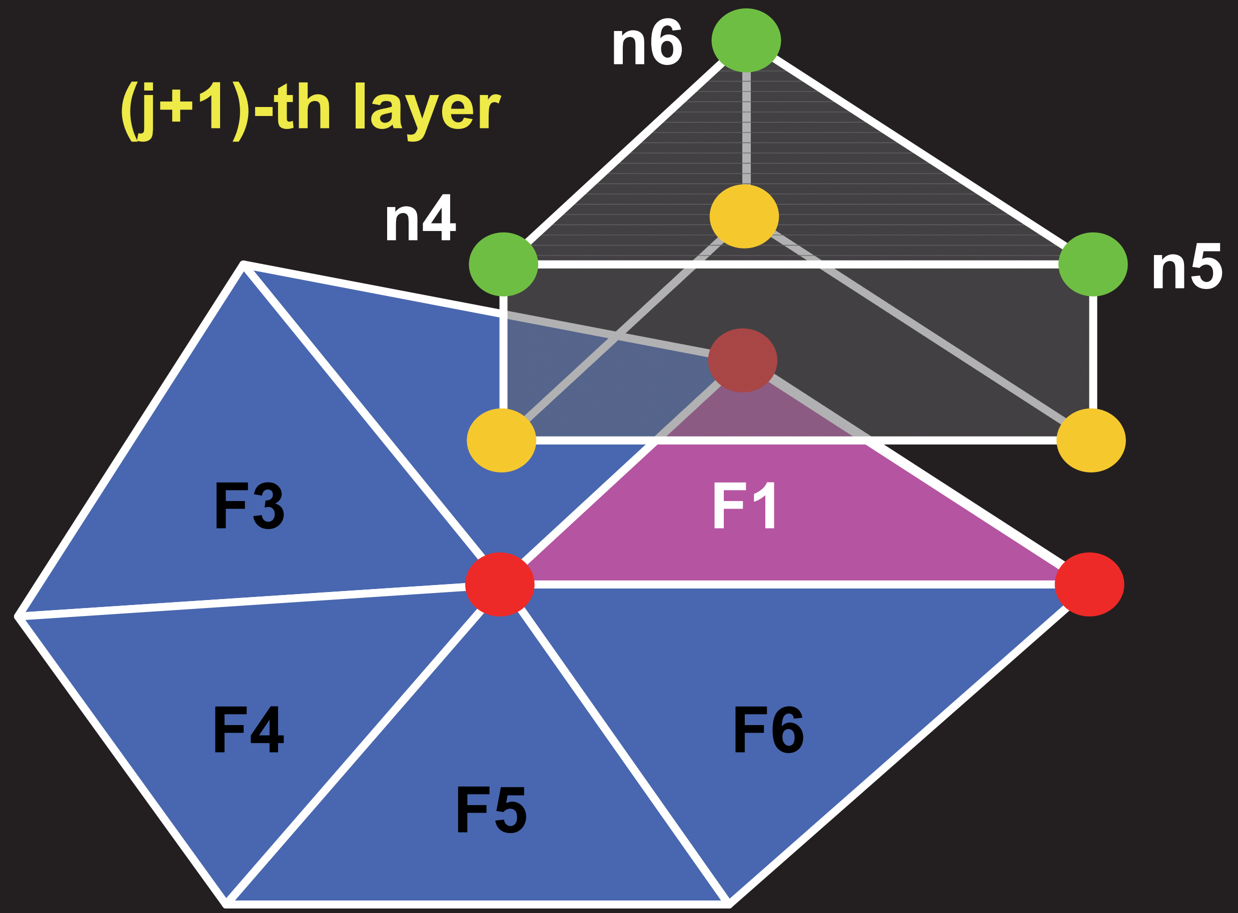
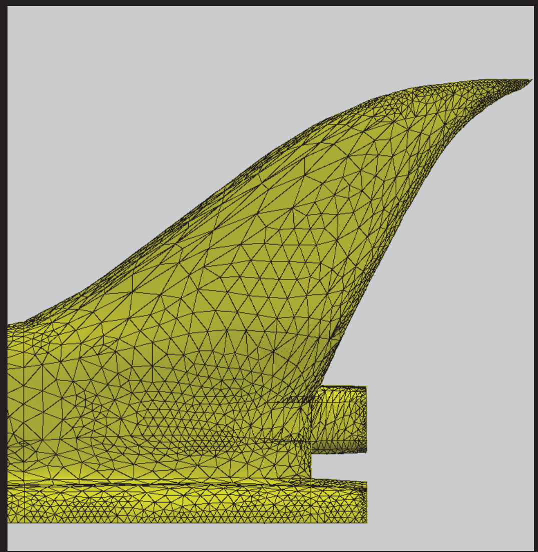
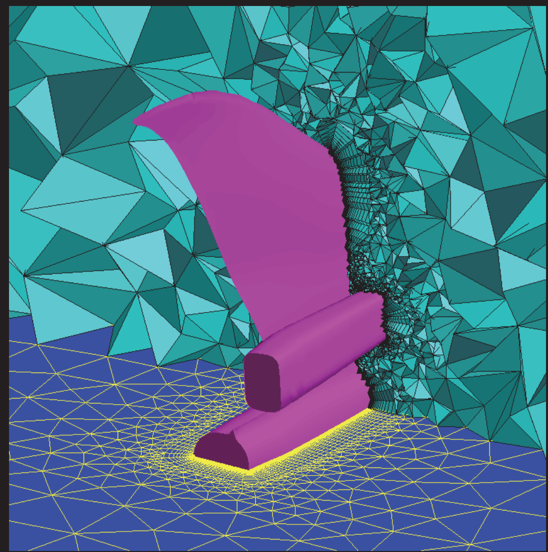
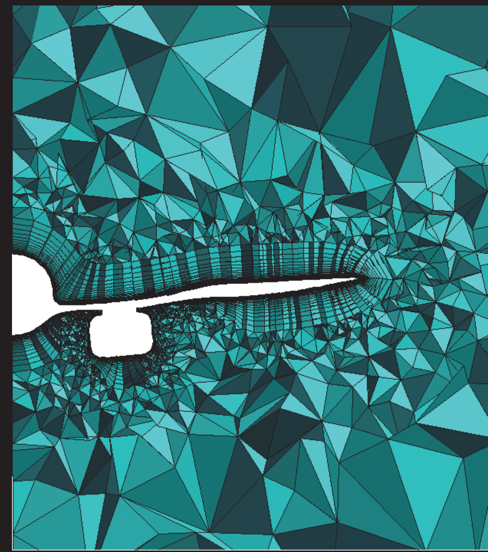


Hybrid Grids
Prisms + Tetrahedra



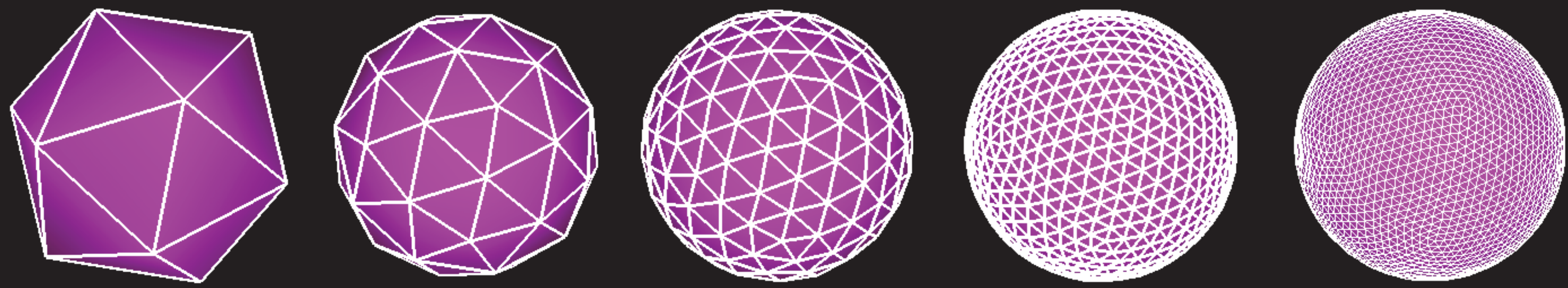




V.Parthasarathy, Y.Kallinderis and K.Nakajima "A Navier-Stokes Method with Adaptive Hybrid Prismatic/Tetrahedral Grids", AIAA Paper 95-0670, 1995.
Y.Kallinderis, A.Khawaja, and H.McMorris "Hybrid Prismatic/Tetrahedral Grid Generation for Complex Geometries", AIAA Paper 95-0211, 1995.
http://diana.ae.utexas.edu/gallery/

Start from Icosahedron

Surface Grid by Adaptive Mesh Refinement (AMR)

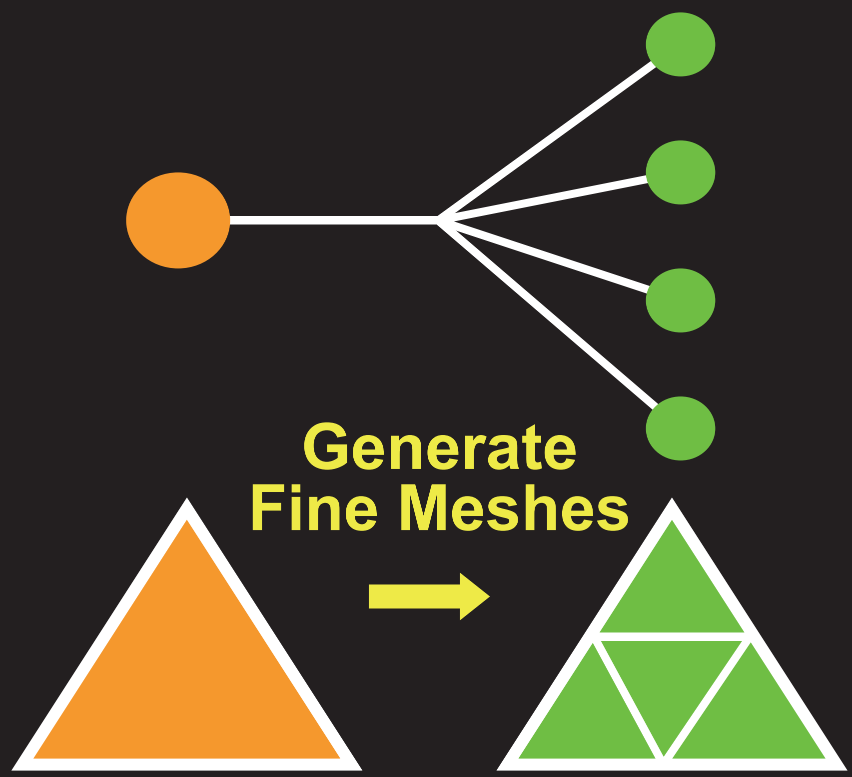


Level 0	Level 1	Level 2	Level 3	Level 4
12 nodes	42 nodes	162 nodes	642 nodes	2,562 nodes
20 tri's	80 tri's	320 tri's	1,280 tri's	5,120 tri's

Refinement of Triangle

Refinement of Triangle

1 parent - 4 children



Refinement of Triangle

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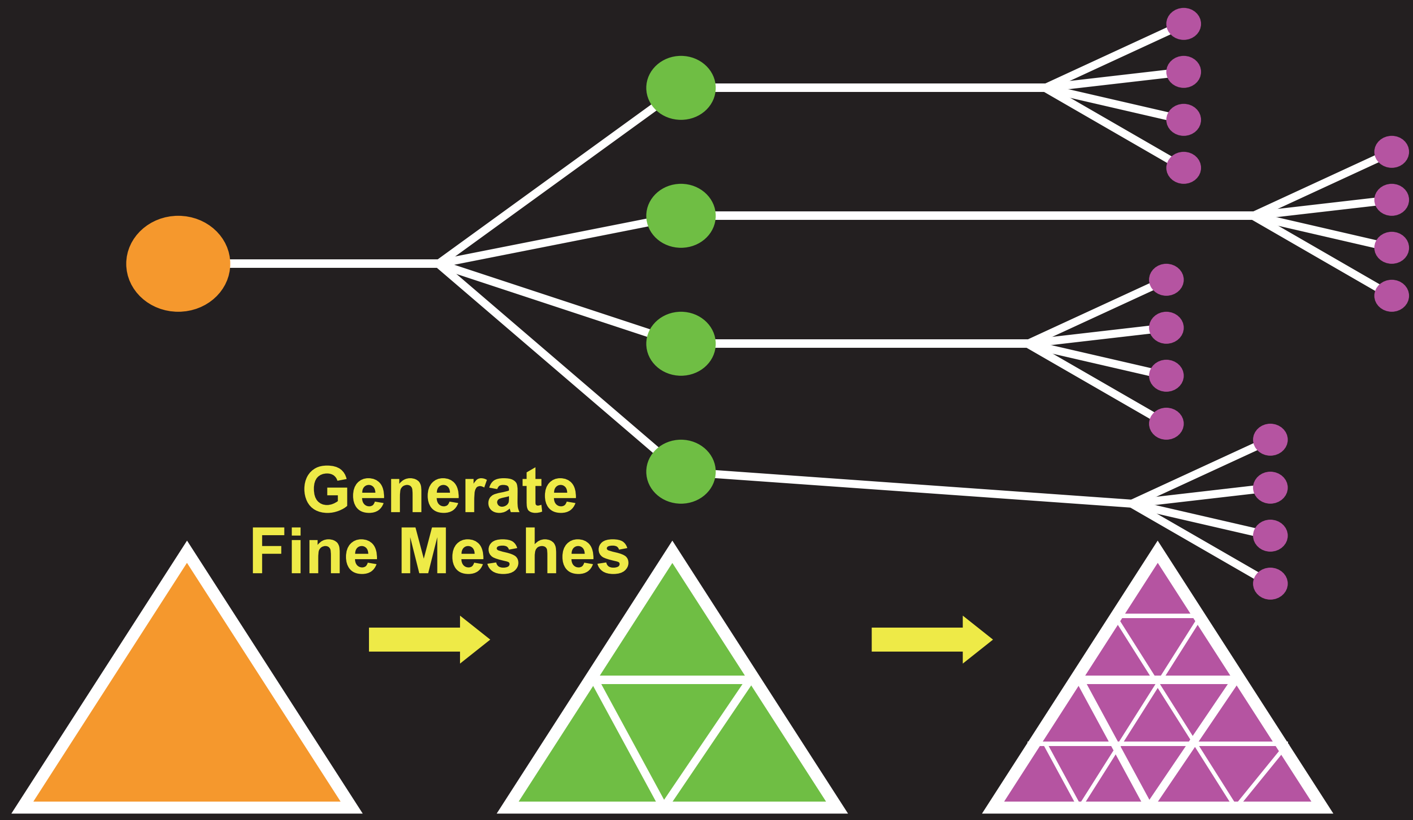
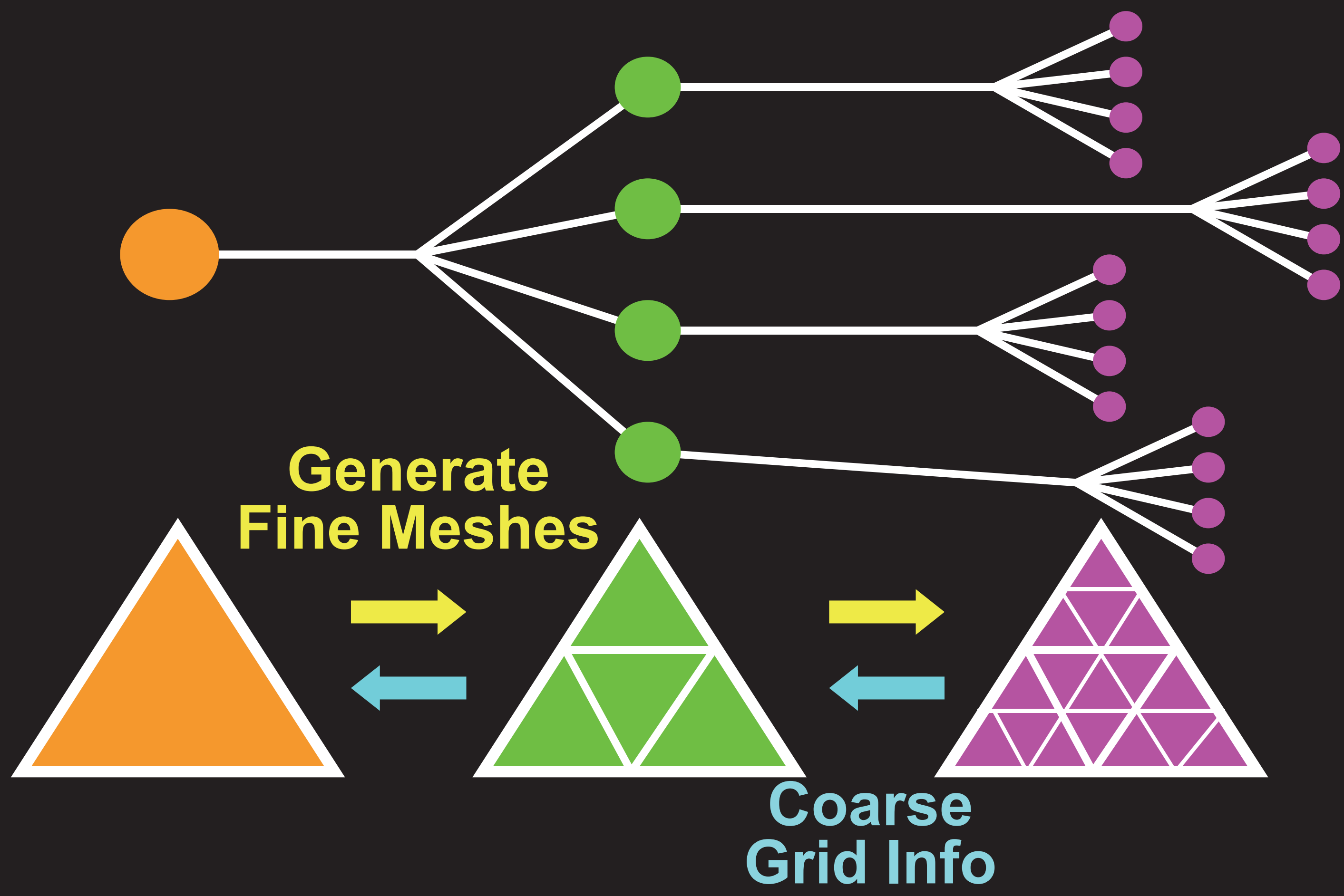


Refinement of Triangle

Info. of grid hierarchy structure is useful for coarsening in multigrid & vistualization



Partitioning
ONLY in RADIAL direction

- NO partition on surface
- Just in order to avoid multilevel load balancing problem during grid adaptation
- Very severe condition for localized ICCG

- Target Application
  - Governing Equations, Method
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  - V-Cycle Multigrid
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  - Large-Scale Examples
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Parallel Multigrid Method

- Preconditioner for CG Solvers
- V-Cycle
- Gauss-Seidel/ILU(0) Smoothers
- Actually, Parallel+Serial (Parallel, then Serial)
  - begin with distributed data by parallel computation
  - Serial computation after levels of coarsening
- Semi-Coarsening in Normal-to-Surface/Lateral Direction Respectively for Very Thin Boundary Layer Meshes
- Multilevel Communication Table
- FORTRAN90 + MPI

Multilevel Method

Multigrid Preconditioning for CG Solver

Compute $r^{(0)} = b - Ax^{(0)}$ for initial guess

for i= 1, 2, …

  solve $Mz^{(i-1)} = r^{(i-1)T}$ → **Multigrid Preconditioning**

  $\rho_{i-1} = r^{(i-1)T} z^{(i-1)}$

  if i= 1

    $p^{(1)} = z^{(0)}$

  else

    $\beta_{i-1} = \rho_{i-1} / \rho_{i-2}$

    $p^{(i)} = z^{(i-1)} + \beta_{i-1} p^{(i-1)}$

  endif

  $q^{(i)} = Ap^{(i)}$

  $\alpha_i = \rho_{i-1} / p^{(i)T} q^{(i)}$

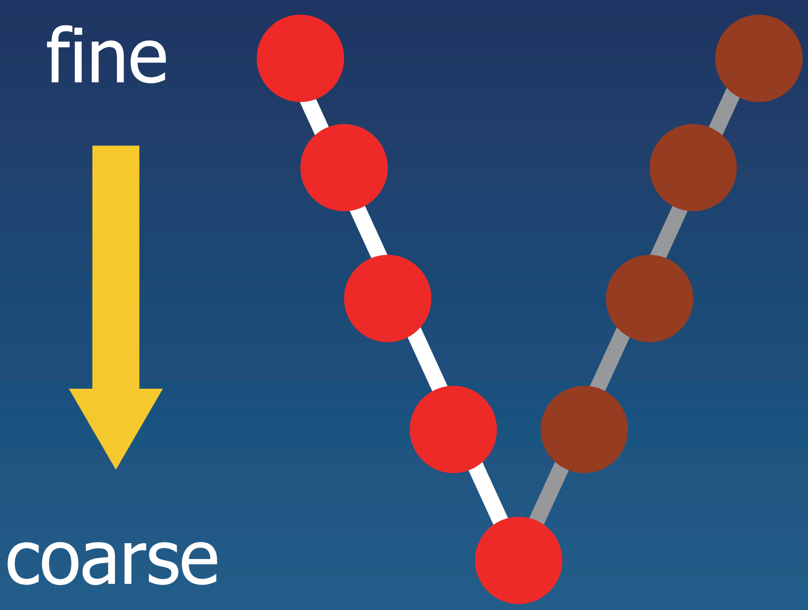
  $x^{(i)} = x^{(i-1)} + \alpha_i p^{(i)}$

  $r^{(i)} = r^{(i-1)} - \alpha_i p^{(i)}$

  check convergence; continue if necessary

end

V-type Multigrid
Restriction



$L^k W^k = F^k$ (Linear Equation: Fine Level)

$R^k = F^k - L^k w_1^k$

$v^k = W^k - w_1^k$, $L^k v^k = R^k$

$R^{k-1} = I_k^{k-1} R^k$

$L^{k-1} v^{k-1} = R^{k-1}$ (Linear Equation: Coarse Level)

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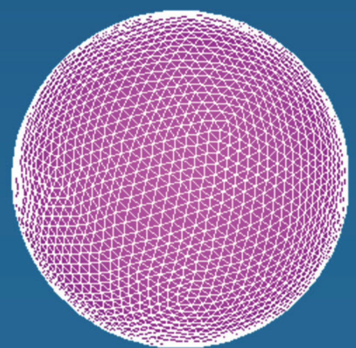
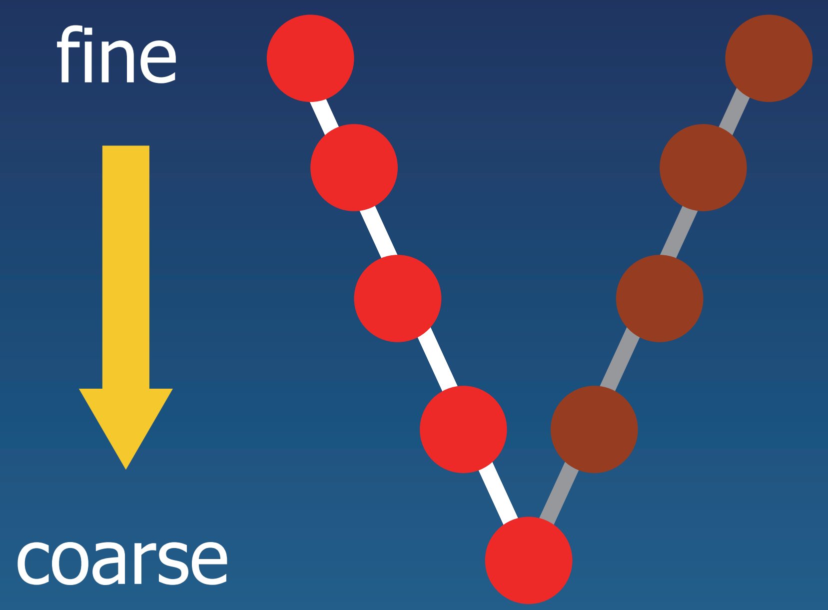
$w_2^k = w_1^k + v^k$



w_1^k : Approx. Solution
v^k : Correction
I_k^{k-1} : Restriction Operator

V-type Multigrid
Restriction



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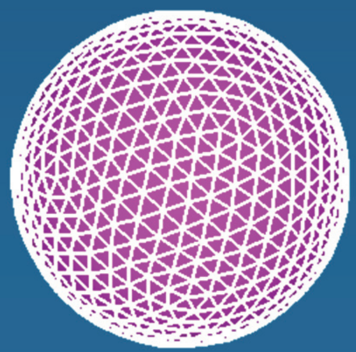

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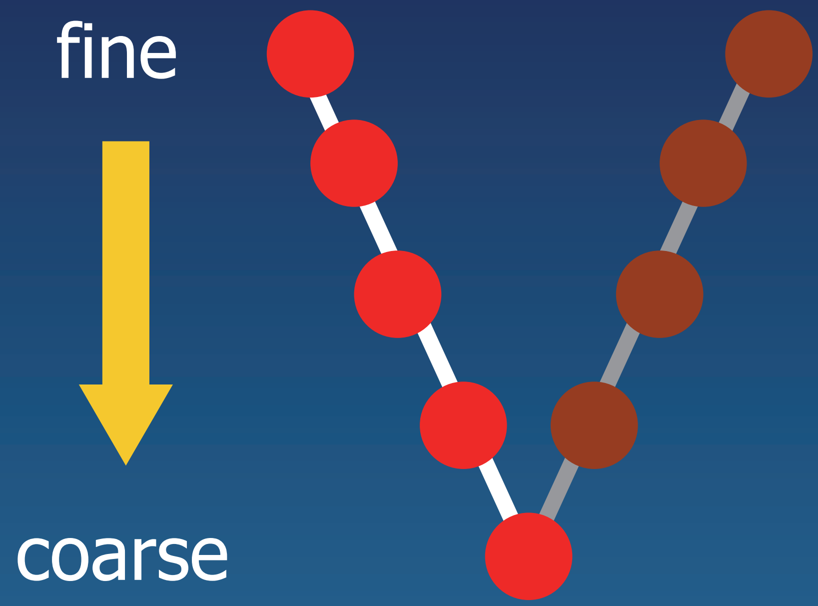

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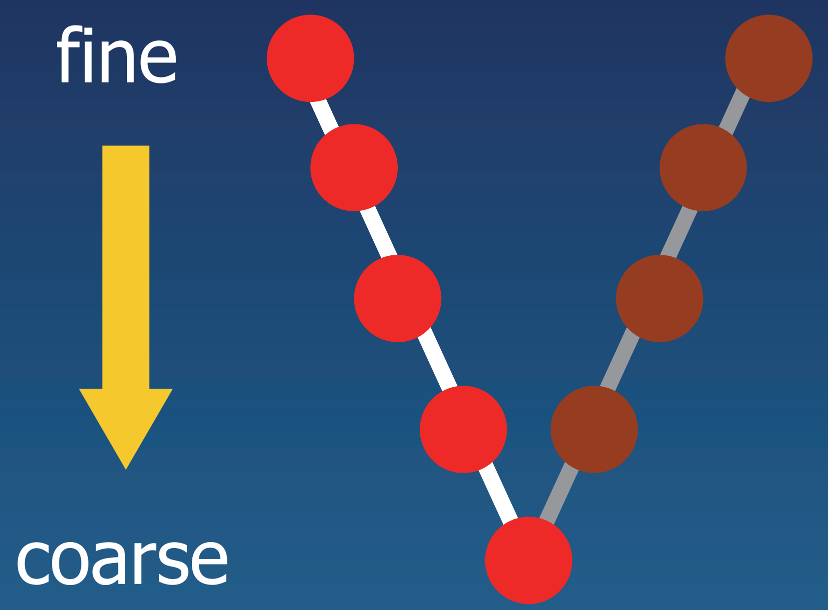
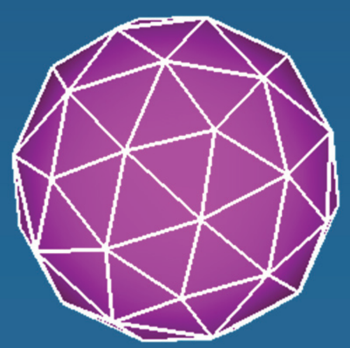
Restriction





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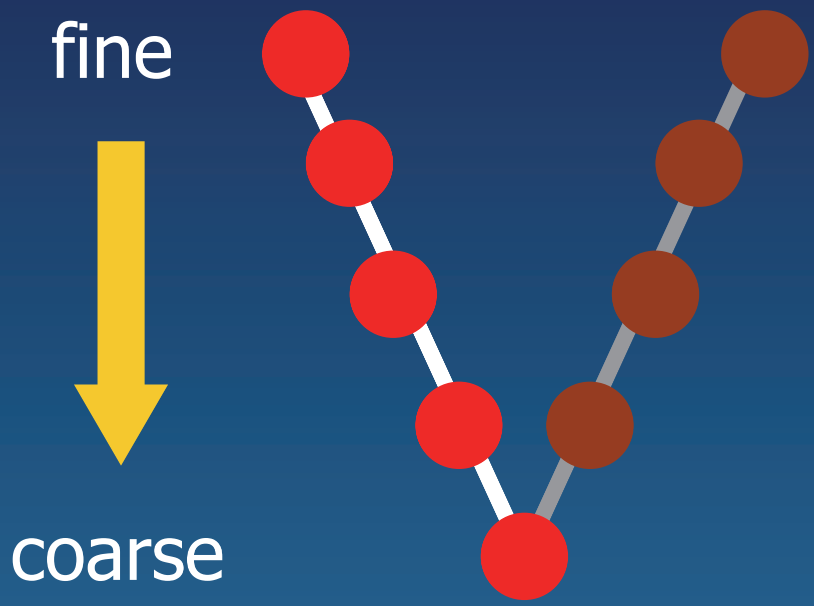
V-type Multigrid
Restriction



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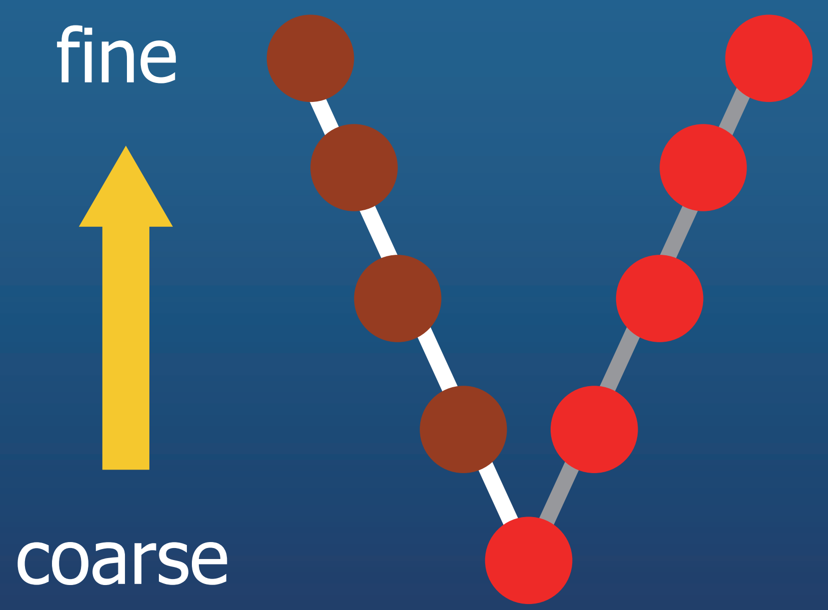
V-type Multigrid
Prolongation



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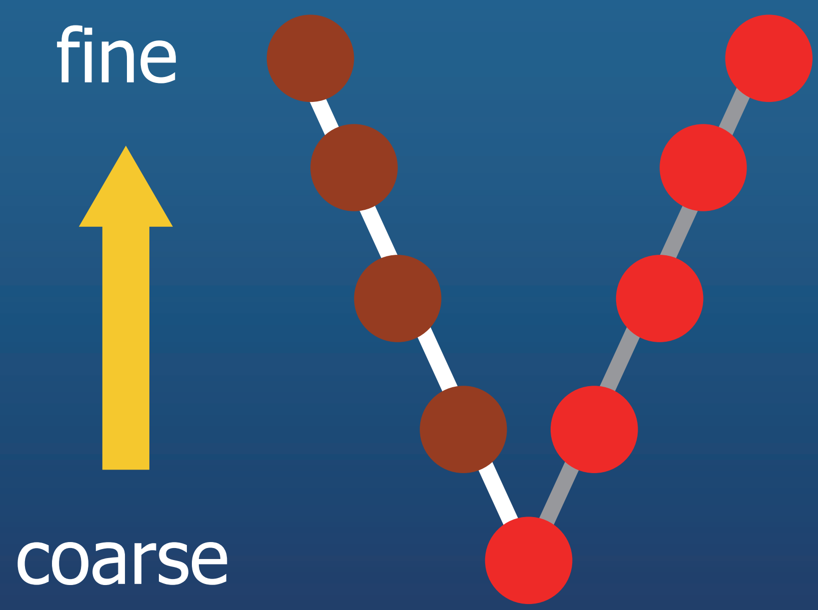
w_2^k : Approx. Solution by Multigrid

V-type Multigrid
Prolongation



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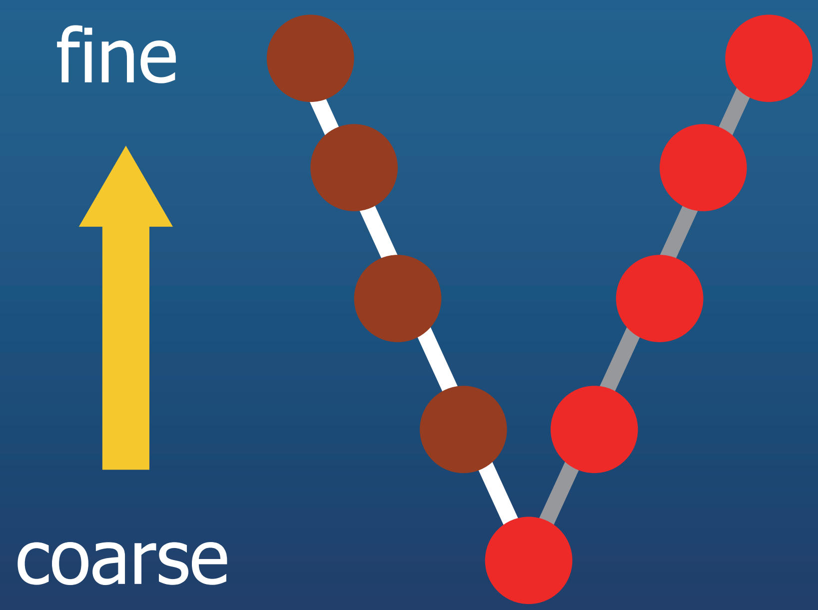
Prolongation



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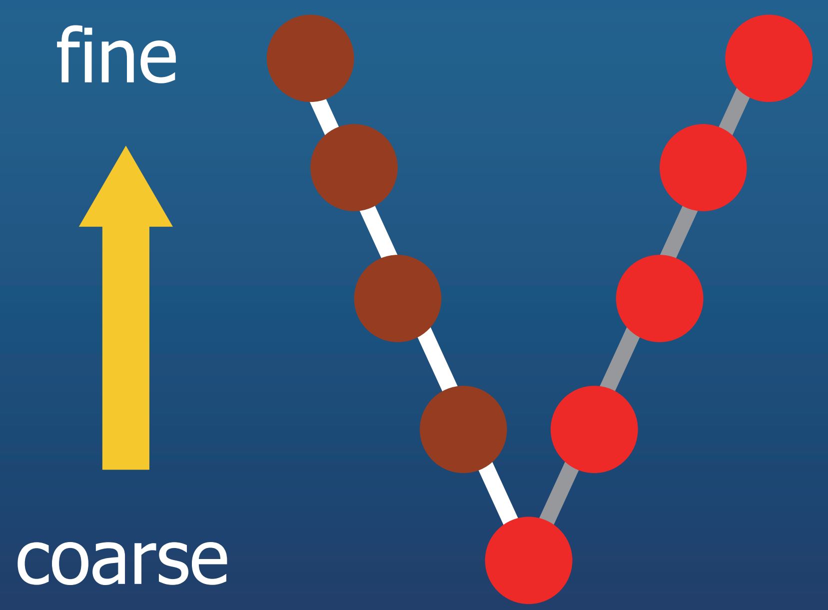
Prolongation



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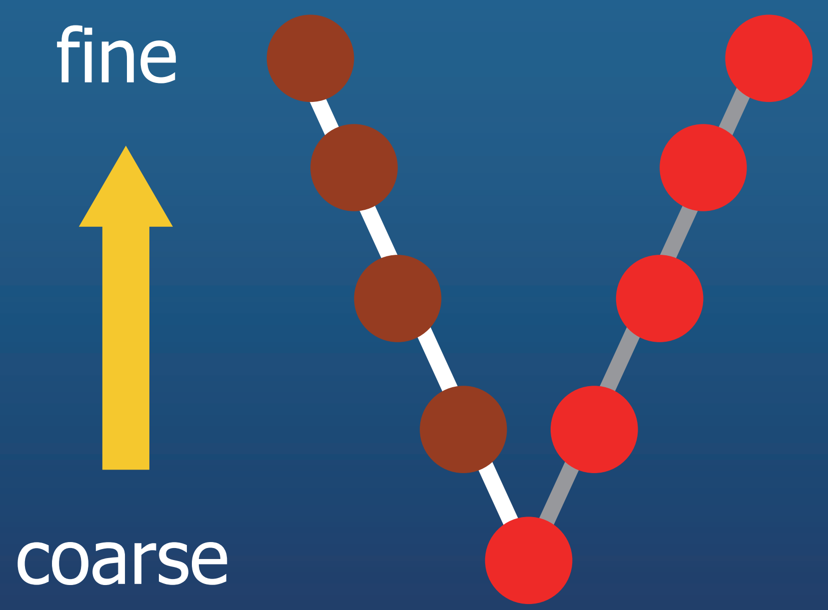

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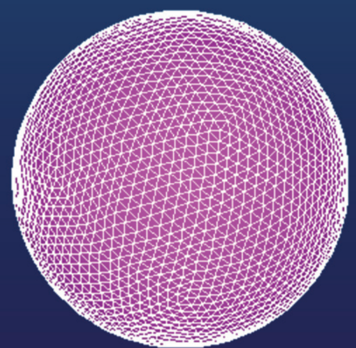

I_{k-1}^k : Prolongation Operator
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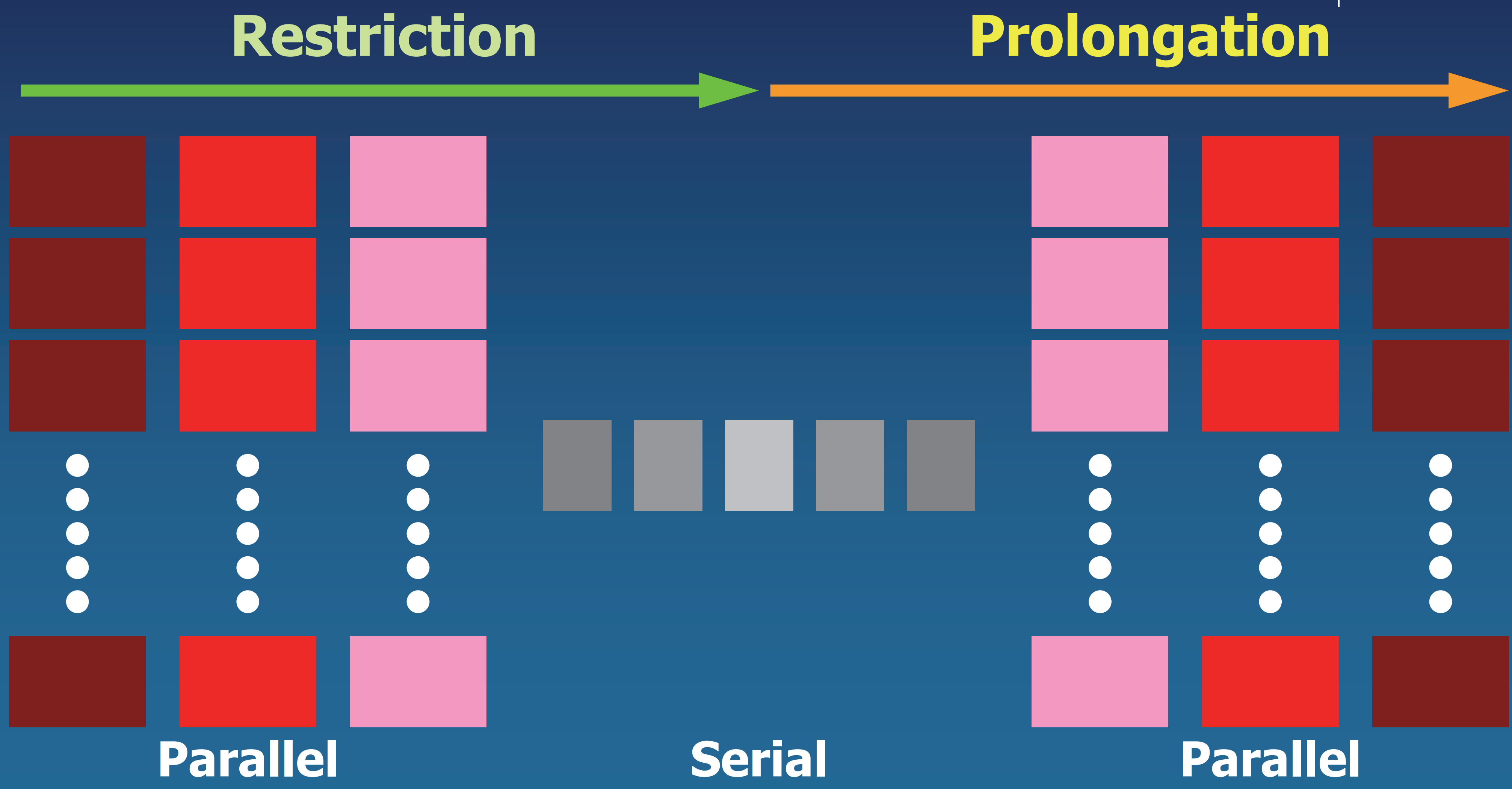
Parallel+Serial Multigrid



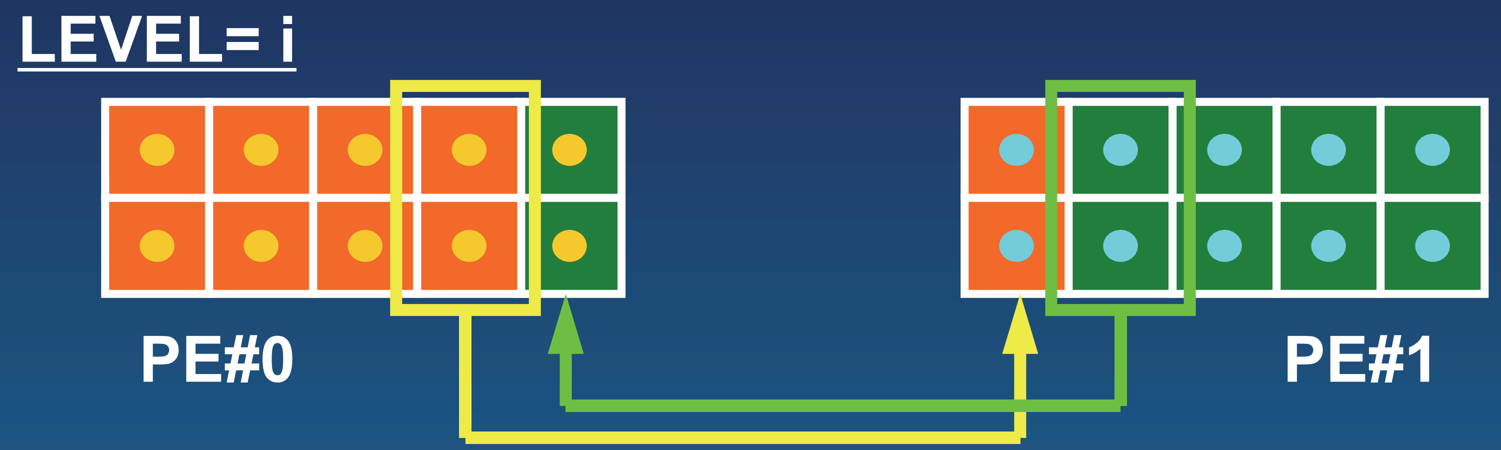
Local Data Structure for Parallel Computing

- ELEMENT-based partitioning
- for EXPLICIT Part of Computation (Momentum/Energy)
  - Interface nodes are shared by neighboring partitions
- for IMPLICIT Part of Computation (Poisson)
  - DOF is defined at cell-center
  - Interface ELEMENTs require information from connected ELEMENTs which are out-of-partitions
- 2-Types of Communication Tables
  - Node & Element-Based, respectively
- Multilevel Communication

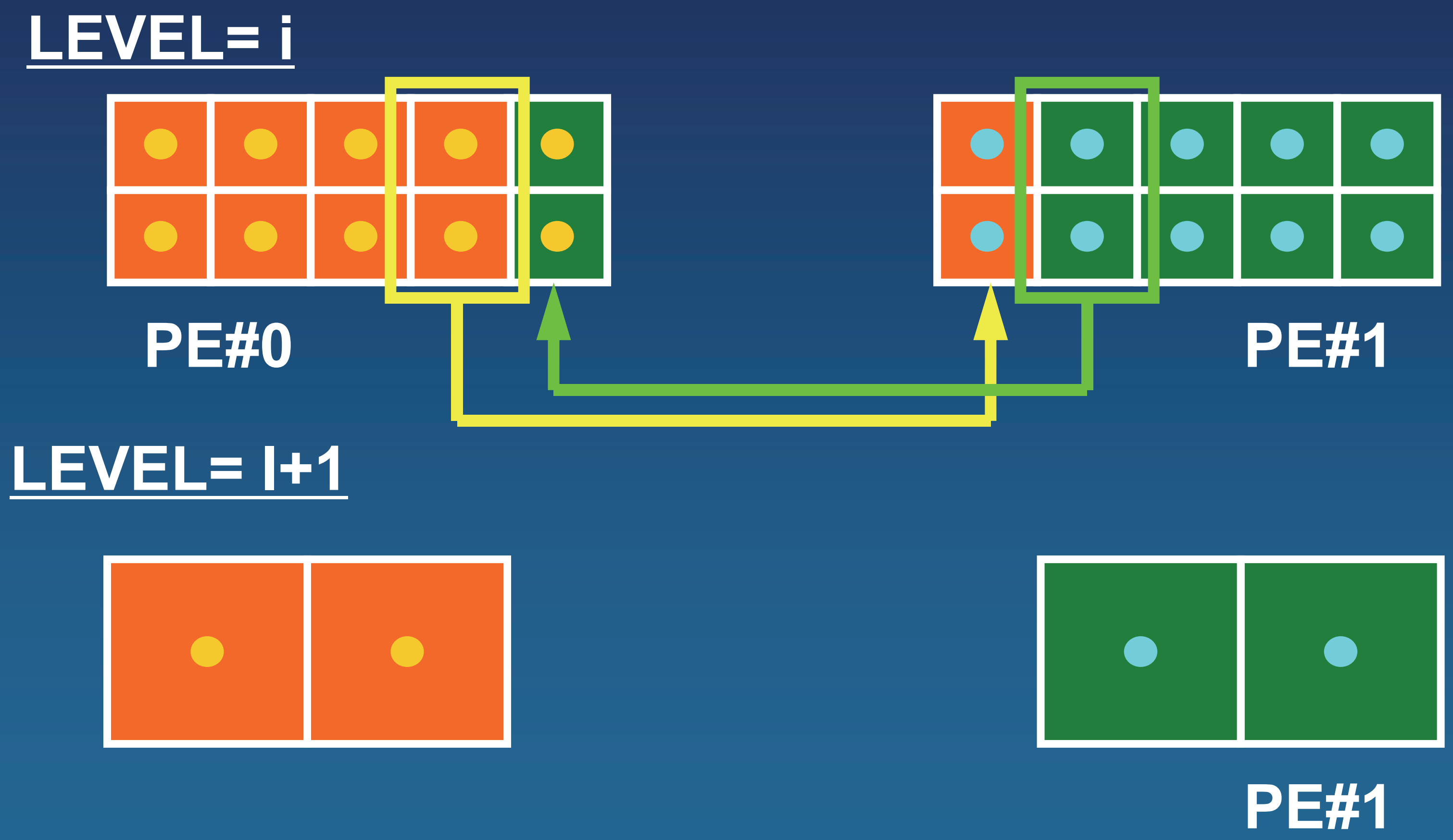
Multilevel Communication Tables using Grid Hierarchy

Multilevel Communication Tables using Grid Hierarchy

Multilevel Communication Tables using Grid Hierarchy



Multilevel Communication Tables using Grid Hierarchy



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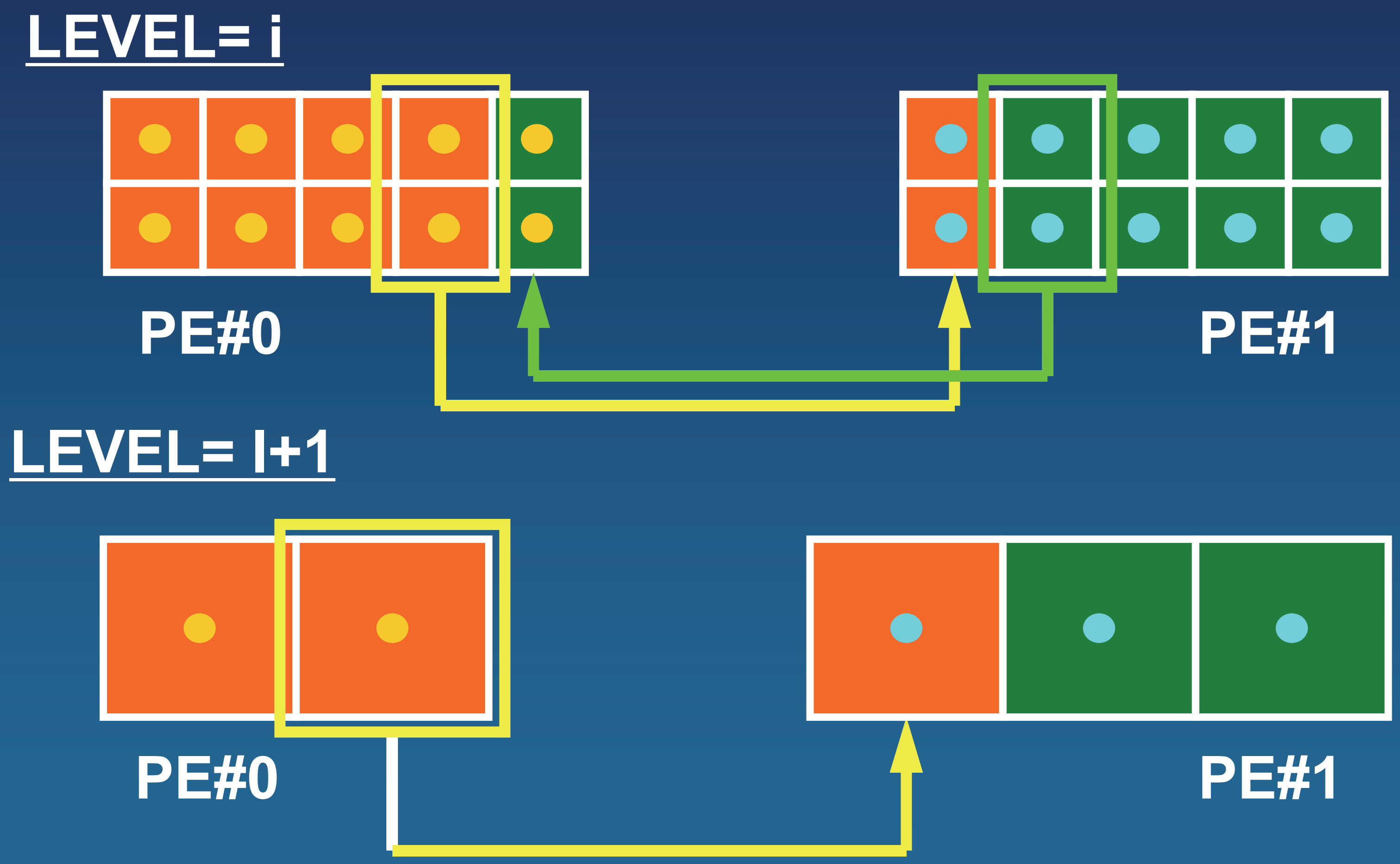
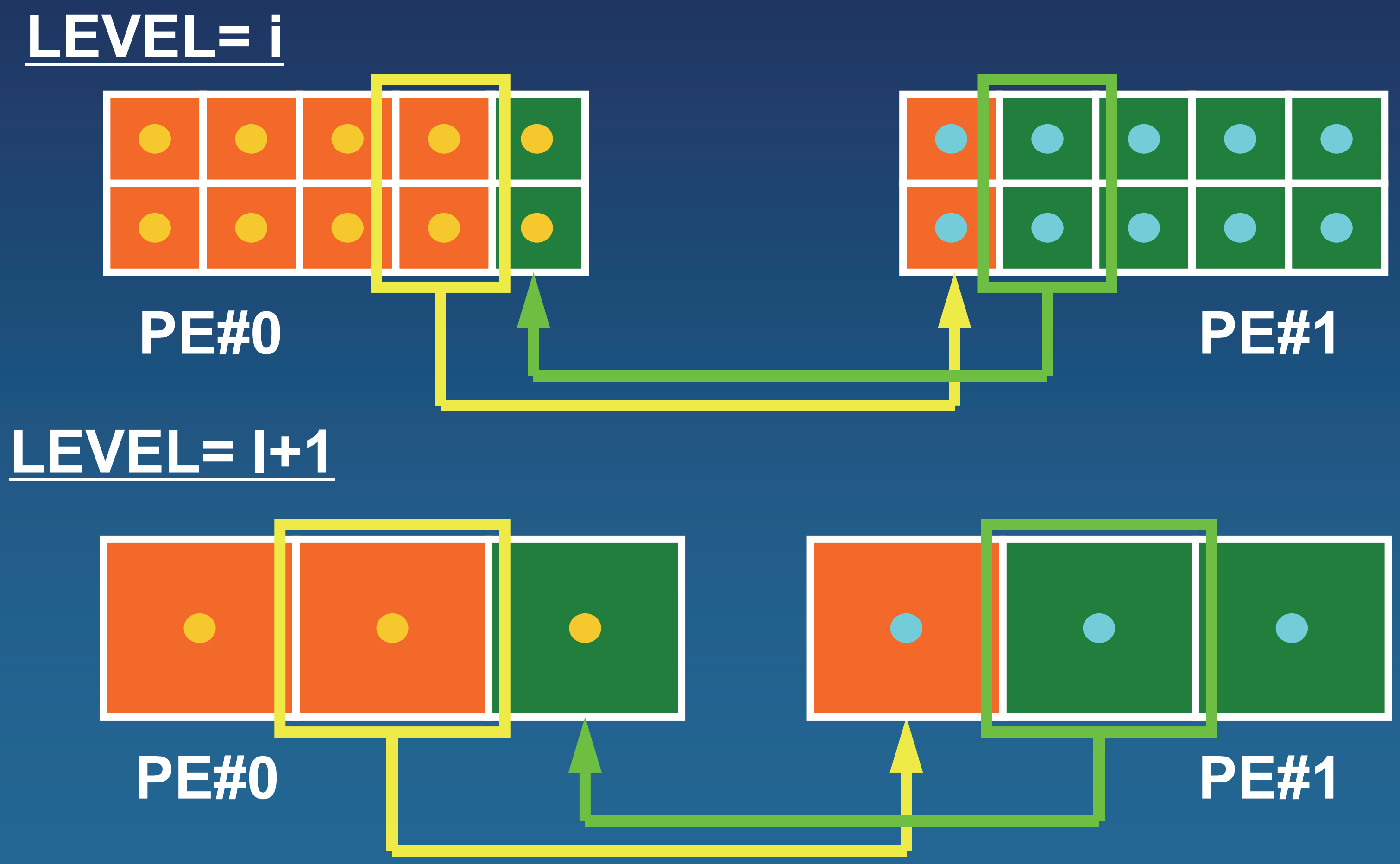


Multilevel Communication Tables using Grid Hierarchy



Multilevel Communication Tables: Operations

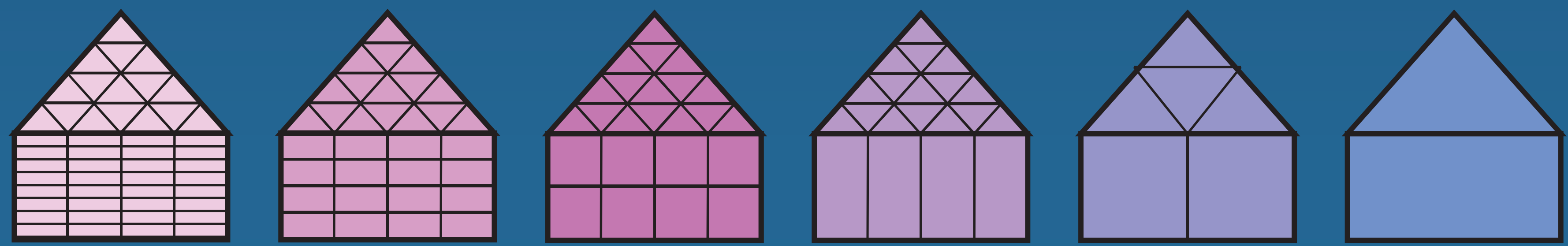
- IMPORT/EXPORT table for corresponding LEVEL should be prepared
- Neighboring PE's can be changed according to the LEVEL

EXAMPLE: lev= LEVEL,SEND BUFFER

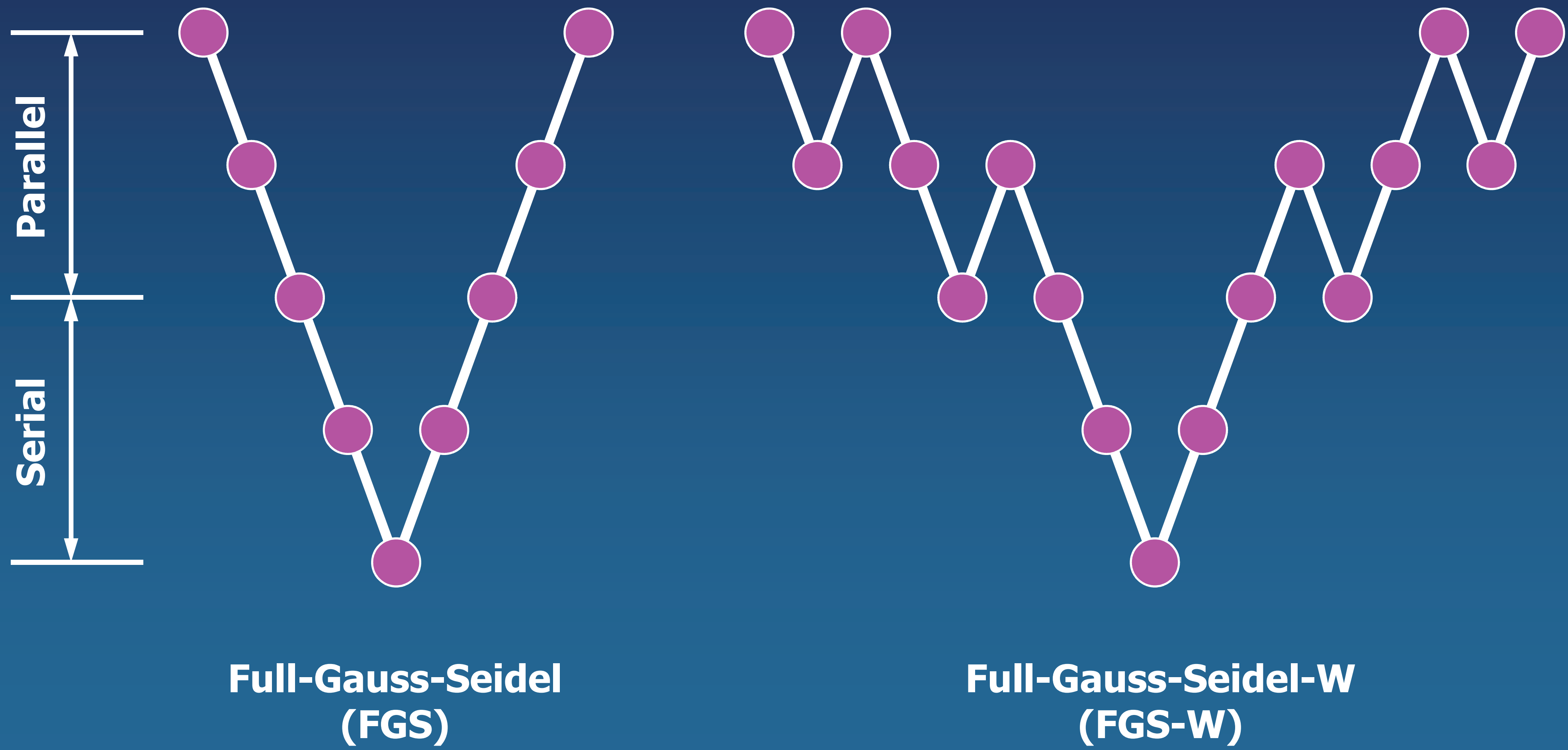
```
NEIBPEstart= NEIBPEindex(lev-1)+1
NEIBPEend  = NEIBPEindex(lev)

do neib= NEIBPEstart, NEIBPEend
  PE= NEIBPE(neib)
  SENDBUFstart= EXPORTindex(neib-1)+1
  SENDBUFend  = EXPORTindex(neib)
  SENDBUFnum  = EXPORTindex(neib)-EXPORTindex(neib-1)
  do k= SENDBUFstart, SENDBUFend
    SENDBUF(k)= VAR(EXPORTitem(k))
  enddo
  call MPI_ISEND (arg.)
enddo
```

Semicoarsening

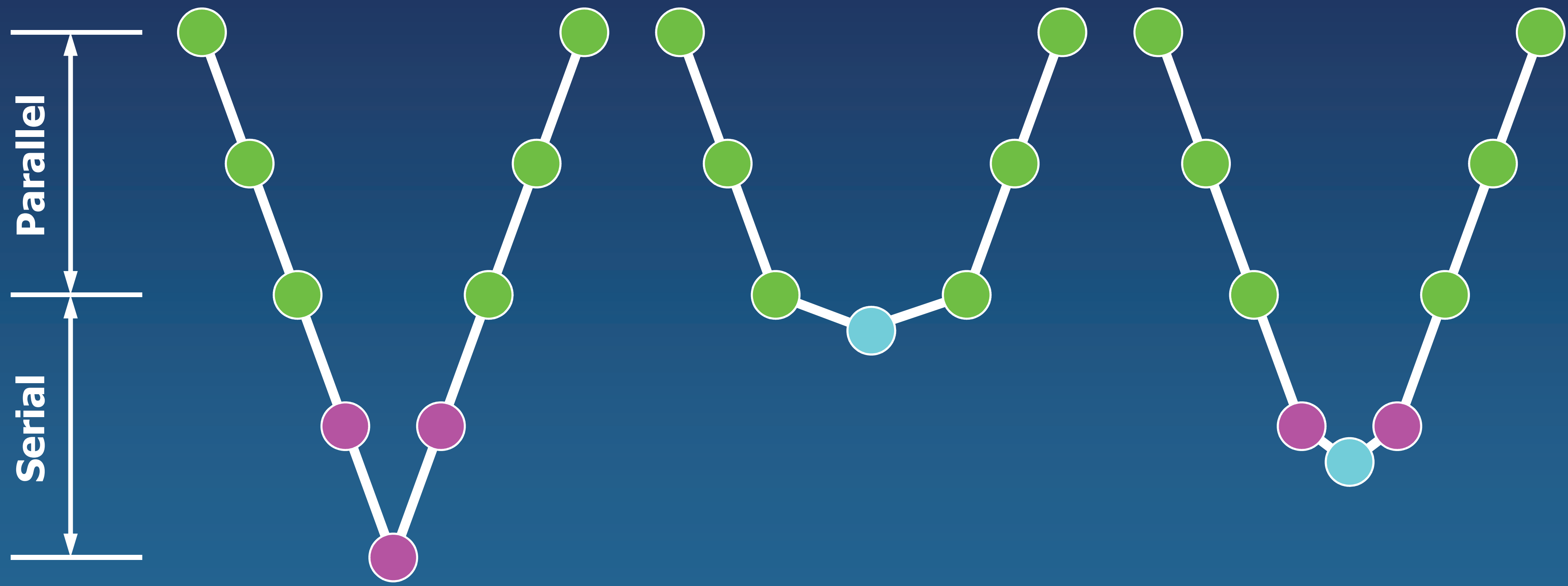
- In lateral and layer (normal-to-surface) direction
  - lateral : one level at one semicoarsening step
  - layer (normal-to-surface) : 2 or 3 layers are merged in one semicoarsening step
  - prolongation process is just injection from the coarsegrid results
- Strategy
  - In V-cycle (restriction), semicoarsening begins with in LAYER direction. If the number of layer in coarsegrid reaches 1 at each processor, then do coarsening in lateral direction.
  - In prolongation process, vice versa



Various Types of Methods - I



Various Types of Methods - II



ILU-Gauss-Seidel
(ILU-GS)

ILU-Serial
Gauss-Seidel-Serial
(ILUs/GSs)

ILU-Gauss-Seidel
-depth=n
(ILU-GS(n))

Serial Solvers on Serial PE

- Gauss-Seidel/SOR method
- 20xPE unknowns
  - It takes longer until convergence if PE# is larger.
  - Entire parallel performance decreases if PE# is larger.
- Control
  - Residual.
  - Number of Iterations.

- Target Application
  - Governing Equations, Method
  - Geometry
  - Mesh Generation by Adaptive Mesh Refinement (AMR)
- Parallel Multigrid Procedure
  - V-Cycle Multigrid
  - Data Structure
- Examples
  - Large-Scale Examples
  - Effect of Local Refinement
- Future Study

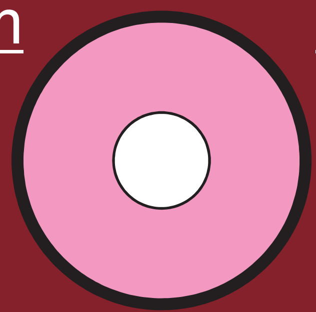
Examples

- Hitachi SR2201 at University of Tokyo
  - 1,024 PEs, 300 MFLOPS peak
  - 256 GB total memory
- Just evaluate the performance of Poisson solvers for large-scale problems
  - Large-Scale Examples
  - Effect of Local Refinement

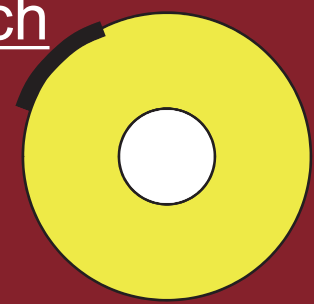
Large-Scale Examples

- Smoothers in MGCG
  - Gauss-Seidel (GS)
  - ILU(0)
- Constant thickness of prisms: Δr=0.01
- Constant RHS force vector in Poisson equation: Δf= 1.00
- Boundary Conditions
  - Uniform Dirichlet BC. on outer sphere surface
  - Dirichlet BC. on just ONE outer initial triangle (one of the 20)

Uniform



Single Patch

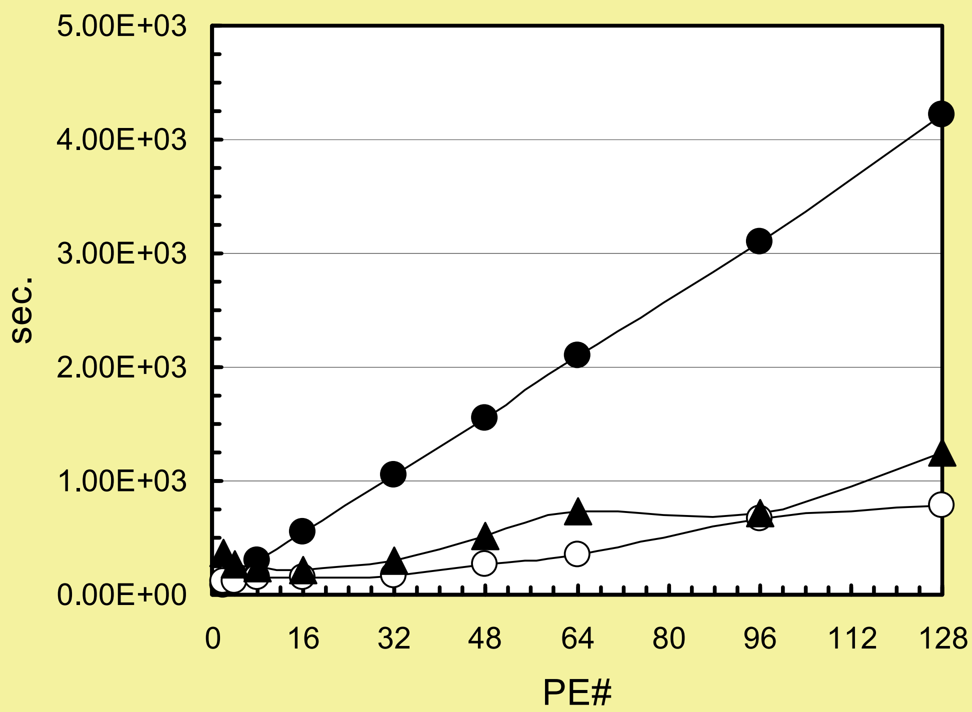


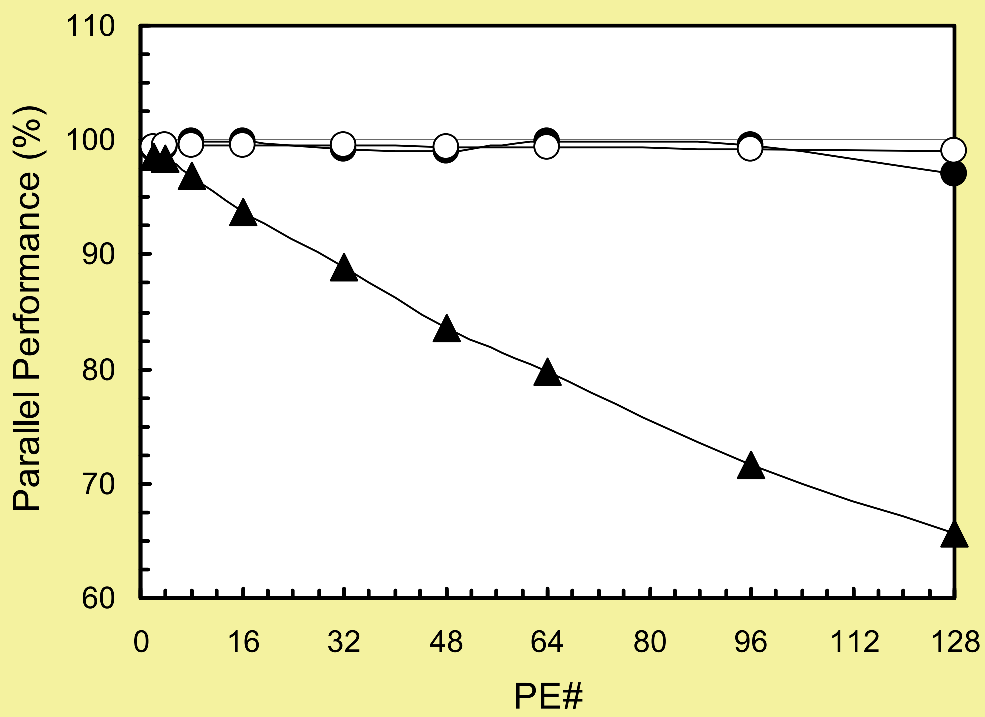
Large-Scale Examples (cont.)

- ICCG
  - Effect of ASDD (Additive Schwartz Domain Decomposition).
- FGS (Full Gauss-Seidel)
  - Effect of Restriction/Prolongation Parameters.
  - Comparison with GSs (GS-Serial).
- ILUs (ILU-Serial)
  - Effect of Iteration Number for Serial Process.
- ILU-GS (ILU-Gauss-Seidel)
  - Comparison with ILUS, ILU-GS(n).

Results for Uniform Boundary Condition

320x900=288,000 cells/PE
up to 37M DOF on 128 PEs





● ICCG ○ MGCG/FGS ▲ MGCG/ILUs

Results for One-Patch Boundary Condition

320x900=288,000 cells/PE
up to 37M DOF on 128 PEs