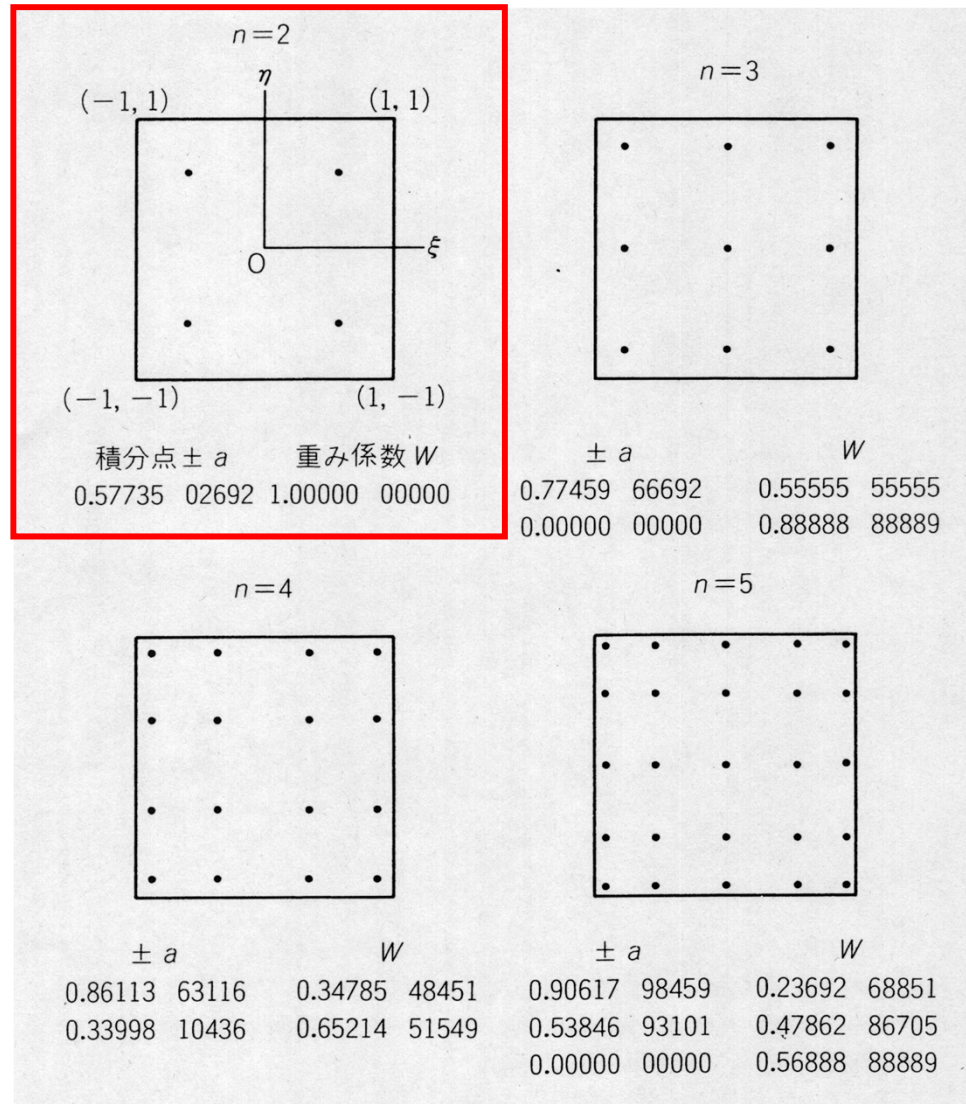


ガウスの積分公式

Gaussian quadrature

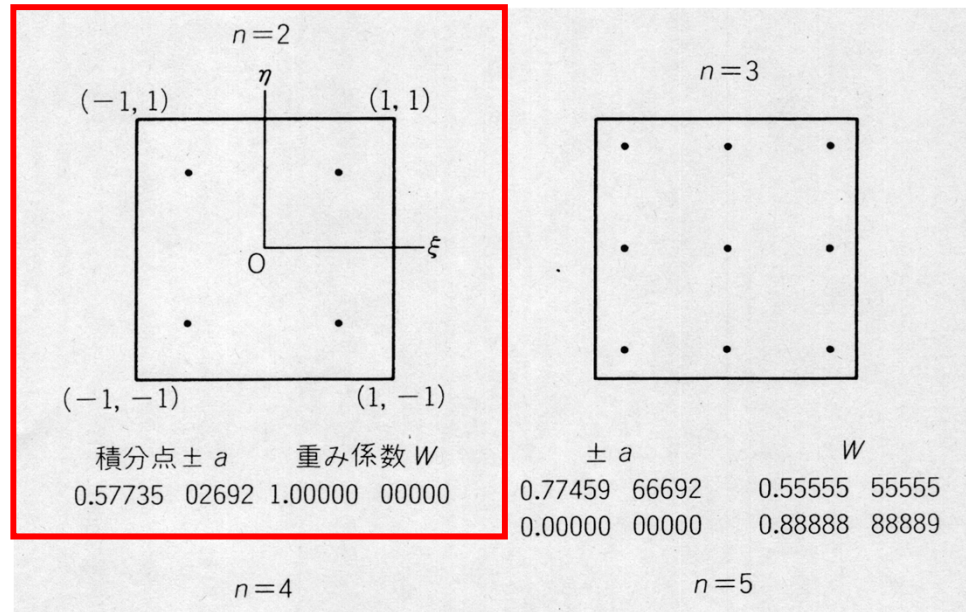
この組み合わせがよく使用される。4点における f の値を使用して領域における積分の値を近似する。



ガウスの積分公式

Gaussian quadrature

この組み合わせがよく使用される。4点におけるfの値を使用して領域における積分の値を近似する。



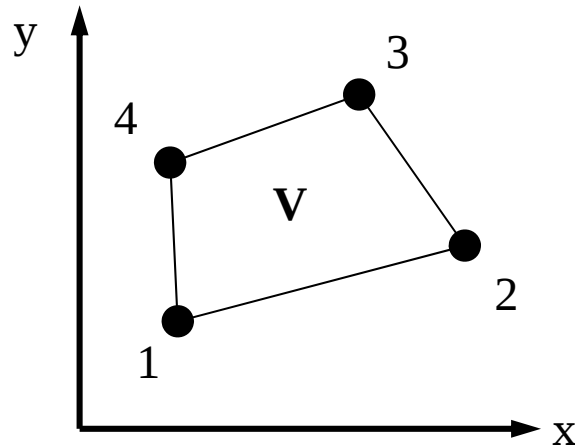
$$I = \int_{-1}^{+1} \int_{-1}^{+1} f(\xi, \eta) d\xi d\eta = \sum_{i=1}^m \sum_{j=1}^n [w_i \cdot w_j \cdot f(\xi_i, \eta_j)]$$

$$= 1.0 \times 1.0 \times f(-0.57735, -0.57735) + 1.0 \times 1.0 \times f(-0.57735, +0.57735) \\ + 1.0 \times 1.0 \times f(+0.57735, +0.57735) + 1.0 \times 1.0 \times f(+0.57735, -0.57735)$$

0.33998 10436 0.65214 51549 0.55846 55101 0.47802 88703
0.00000 00000 0.56888 88889

宿題

- ガウスの積分公式を使用して以下の四角形の面積を求めよ



1: (1.0, 1.0)
2: (4.0, 2.0)
3: (3.0, 5.0)
4: (2.0, 4.0)

$$I = \int_V dV$$

ヒント

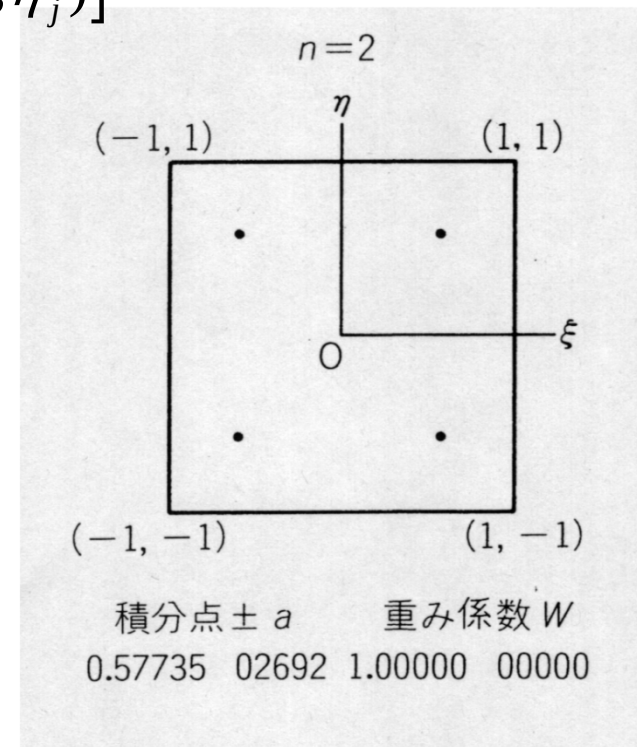
- 座標値によってヤコビアン（ヤコビの行列）を計算する。
- ガウスの積分公式（ $n=2$ ）に代入する。

$$I = \int_{-1}^{+1} \int_{-1}^{+1} f(\xi, \eta) d\xi d\eta = \sum_{i=1}^m \sum_{j=1}^n [W_i \cdot W_j \cdot f(\xi_i, \eta_j)]$$

```
implicit REAL*8 (A-H,O-Z)
real*8 W(2)
real*8 POI(2)

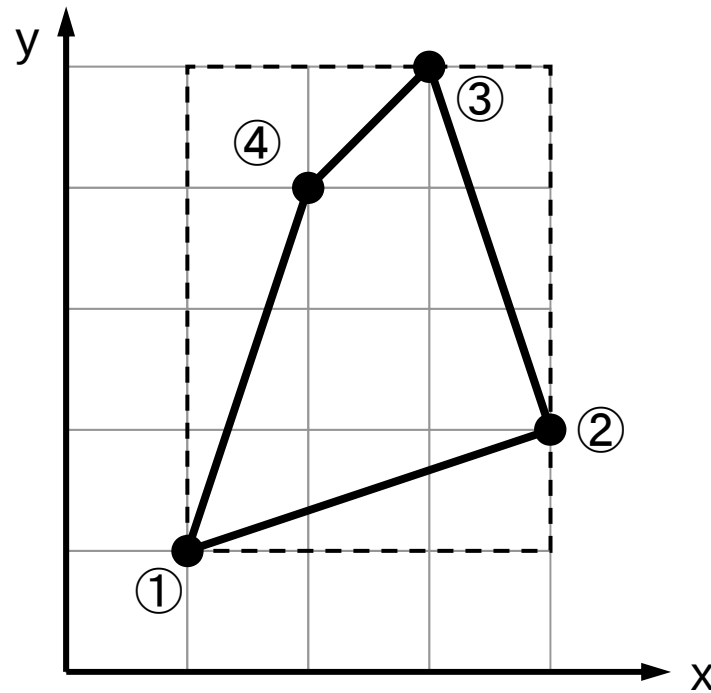
W(1)= 1.0d0
W(2)= 1.0d0
POI(1)= -0.5773502692d0
POI(2)= +0.5773502692d0

SUM= 0.d0
do jp= 1, 2
do ip= 1, 2
    FC = F(POI(ip),POI(jp))
    SUM= SUM + W (ip)*W (jp)*FC
enddo
enddo
```



正解

1: (1.0, 1.0)
2: (4.0, 2.0)
3: (3.0, 5.0)
4: (2.0, 4.0)



から四隅の三角形を取り除く

$$3 \times 4 - \frac{1}{2}(3 + 3 + 2 + 4) \times 1$$
$$= 12 - \frac{12}{2} = 6$$

何をすべきか？

- 以下の積分を求めればよい：

$$I = \int_V dV = \iint dx dy = \int_{-1}^{+1} \int_{-1}^{+1} \det[J] d\xi d\eta$$

- 定義によれば：

$$I = \int_{-1}^{+1} \int_{-1}^{+1} f(\xi, \eta) d\xi d\eta = \sum_{i=1}^m \sum_{j=1}^n [W_i \cdot W_j \cdot f(\xi_i, \eta_j)]$$

- つまり， $\det [J] = f$
- **ということは $\det [J]$ の積分点における値を求めれば良い！**

$$\det[J(\xi_i, \eta_j)]$$

初期設定 (1/4)

```
implicit REAL*8 (A-H,O-Z)

real*8 X(4), Y(4)
real*8 W(2), POS(2)
real*8 SHAPE(2,2,4)
real*8 PNQ(2,4), PNE(2,4), DETJ(2,2)
```

```
!C
!C-- POINT data
X(1)= 1.0
Y(1)= 1.0
```

```
X(2)= 4.0
Y(2)= 2.0
```

```
X(3)= 3.0
Y(3)= 5.0
```

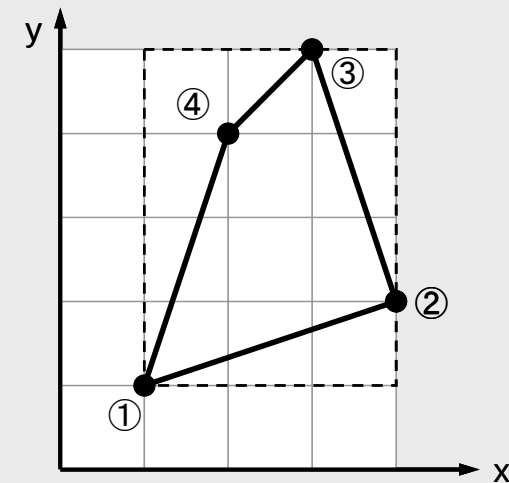
```
X(4)= 2.0
Y(4)= 4.0
```

```
!C
!C-- Quadrature points & weighting coef.
W(1)= +1.000000000000d+00
W(2)= +1.000000000000d+00
```

```
POS(1)= -0.5773502692d+00
POS(2)= +0.5773502692d+00
```

各節点の座標

```
1: (1.0, 1.0)
2: (4.0, 2.0)
3: (3.0, 5.0)
4: (2.0, 4.0)
```



初期設定 (1/4)

```
implicit REAL*8 (A-H,O-Z)

real*8 X(4), Y(4)
real*8 W(2), POS(2)
real*8 SHAPE(2,2,4)
real*8 PNQ(2,4), PNE(2,4), DETJ(2,2)
```

```
!C
!C-- POINT data
      X(1)= 1.0
      Y(1)= 1.0
```

```
      X(2)= 4.0
      Y(2)= 2.0
```

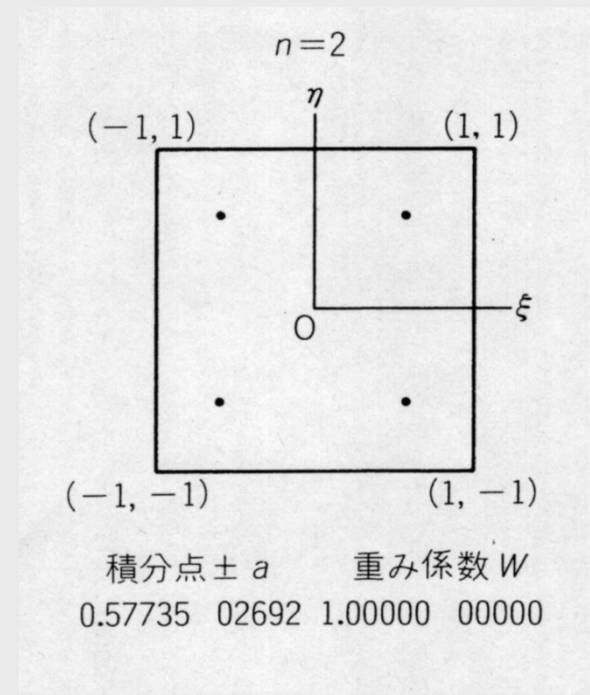
```
      X(3)= 3.0
      Y(3)= 5.0
```

```
      X(4)= 2.0
      Y(4)= 4.0
```

POS: 積分点座標
W: 重み係数

```
!C
!C-- Quadrature points & weighting coef.
      W(1)= +1.0000000000d+00
      W(2)= +1.0000000000d+00
```

```
      POS(1)= -0.5773502692d+00
      POS(2)= +0.5773502692d+00
```



積分点における形状関数, その微分 (2/4)

```
!C
!C-- SHAPE functions
      04th= 0.25d0
```

```
do jp= 1, 2
do ip= 1, 2
  QP1= 1.d0 + POS(ip)
  QM1= 1.d0 - POS(ip)
  EP1= 1.d0 + POS(jp)
  EM1= 1.d0 - POS(jp)
```

$$QP1(i) = (1 + \xi_i), \quad QM1(i) = (1 - \xi_i)$$

$$EP1(j) = (1 + \eta_j), \quad EM1(j) = (1 - \eta_j)$$

```
SHAPE(ip, jp, 1)= 04th * QM1 * EM1
SHAPE(ip, jp, 2)= 04th * QP1 * EM1
SHAPE(ip, jp, 3)= 04th * QP1 * EP1
SHAPE(ip, jp, 4)= 04th * QM1 * EP1
```

```
PNQ(jp, 1)= - 04th * EM1
PNQ(jp, 2)= + 04th * EM1
PNQ(jp, 3)= + 04th * EP1
PNQ(jp, 4)= - 04th * EP1
```

```
PNE(ip, 1)= - 04th * QM1
PNE(ip, 2)= - 04th * QP1
PNE(ip, 3)= + 04th * QP1
PNE(ip, 4)= + 04th * QM1
```

```
enddo
enddo
```

積分点における形状関数, その微分 (2/4)

```
!C
!C-- SHAPE functions
      04th= 0.25d0
```

```
do jp= 1, 2
do ip= 1, 2
  QP1= 1.d0 + POS(ip)
  QM1= 1.d0 - POS(ip)
  EP1= 1.d0 + POS(jp)
  EM1= 1.d0 - POS(jp)
```

```
  SHAPE(ip,jp,1)= 04th * QM1 * EM1
  SHAPE(ip,jp,2)= 04th * QP1 * EM1
  SHAPE(ip,jp,3)= 04th * QP1 * EP1
  SHAPE(ip,jp,4)= 04th * QM1 * EP1
```

```
  PNQ(jp,1)= - 04th * EM1
  PNQ(jp,2)= + 04th * EM1
  PNQ(jp,3)= + 04th * EP1
  PNQ(jp,4)= - 04th * EP1
```

```
  PNE(ip,1)= - 04th * QM1
  PNE(ip,2)= - 04th * QP1
  PNE(ip,3)= + 04th * QP1
  PNE(ip,4)= + 04th * QM1
```

```
enddo
enddo
```

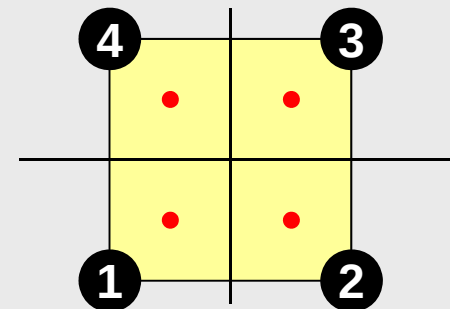
SHAPE: (ξ_i, η_j) における形状関数の値

$$N_1(\xi_i, \eta_j) = \frac{1}{4}(1 - \xi_i)(1 - \eta_j)$$

$$N_2(\xi_i, \eta_j) = \frac{1}{4}(1 + \xi_i)(1 - \eta_j)$$

$$N_3(\xi_i, \eta_j) = \frac{1}{4}(1 + \xi_i)(1 + \eta_j)$$

$$N_4(\xi_i, \eta_j) = \frac{1}{4}(1 - \xi_i)(1 + \eta_j)$$



積分点における形状関数, その微分 (2/4)

```
!C
!C-- SHAPE functions
      04th= 0.25d0
```

```
do jp= 1, 2
do ip= 1, 2
  QP1= 1.d0 + POS(ip)
  QM1= 1.d0 - POS(ip)
  EP1= 1.d0 + POS(jp)
  EM1= 1.d0 - POS(jp)
```

```
SHAPE(ip,jp,1)= 04th * QM1 * EM1
SHAPE(ip,jp,2)= 04th * QP1 * EM1
SHAPE(ip,jp,3)= 04th * QP1 * EP1
SHAPE(ip,jp,4)= 04th * QM1 * EP1
```

```
PNQ(jp,1)= - 04th * EM1
PNQ(jp,2)= + 04th * EM1
PNQ(jp,3)= + 04th * EP1
PNQ(jp,4)= - 04th * EP1
```

```
PNE(ip,1)= - 04th * QM1
PNE(ip,2)= - 04th * QP1
PNE(ip,3)= + 04th * QP1
PNE(ip,4)= + 04th * QM1
```

```
enddo
enddo
```

$$PNQ(j,k) = \frac{\partial N_k}{\partial \xi} (\xi = \xi_i, \eta = \eta_j)$$

$$PNE(j,k) = \frac{\partial N_k}{\partial \eta} (\xi = \xi_i, \eta = \eta_j)$$

$$\frac{\partial N_1}{\partial \xi}(\xi_i, \eta_j) = -\frac{1}{4}(1-\eta_j) \quad \frac{\partial N_1}{\partial \eta}(\xi_i, \eta_j) = -\frac{1}{4}(1-\xi_i)$$

$$\frac{\partial N_2}{\partial \xi}(\xi_i, \eta_j) = +\frac{1}{4}(1-\eta_j) \quad \frac{\partial N_2}{\partial \eta}(\xi_i, \eta_j) = -\frac{1}{4}(1+\xi_i)$$

$$\frac{\partial N_3}{\partial \xi}(\xi_i, \eta_j) = +\frac{1}{4}(1+\eta_j) \quad \frac{\partial N_3}{\partial \eta}(\xi_i, \eta_j) = +\frac{1}{4}(1+\xi_i)$$

$$\frac{\partial N_4}{\partial \xi}(\xi_i, \eta_j) = -\frac{1}{4}(1+\eta_j) \quad \frac{\partial N_4}{\partial \eta}(\xi_i, \eta_j) = +\frac{1}{4}(1-\xi_i)$$

(ξ_i, η_j) における形状関数の一階微分

積分点におけるヤコビアン 計算 (3/4)

```
!C
!C +-----+
!C | JACOBIAN matrix |
!C +-----+
!C===
      do jp= 1, 2
      do ip= 1, 2
        dXdQ = PNQ(jp,1)*X(1) + PNQ(jp,2)*X(2) +
&            PNQ(jp,3)*X(3) + PNQ(jp,4)*X(4) +
        dYdQ = PNQ(jp,1)*Y(1) + PNQ(jp,2)*Y(2) +
&            PNQ(jp,3)*Y(3) + PNQ(jp,4)*Y(4) +
        dXdE = PNE(ip,1)*X(1) + PNE(ip,2)*X(2) +
&            PNE(ip,3)*X(3) + PNE(ip,4)*X(4) +
        dYdE = PNE(ip,1)*Y(1) + PNE(ip,2)*Y(2) +
&            PNE(ip,3)*Y(3) + PNE(ip,4)*Y(4) +
        DETJ(ip,jp)= dXdQ*dYdE - dXdE*dYdQ
      enddo
    enddo
!C===
```

$$dXdQ = \frac{\partial x}{\partial \xi} \quad dYdQ = \frac{\partial y}{\partial \xi}$$

$$dXdE = \frac{\partial x}{\partial \eta} \quad dYdE = \frac{\partial y}{\partial \eta}$$

$$DETJ(i, j) = \det[J(\xi_i, \eta_j)]$$

$$J_{11} = \frac{\partial x}{\partial \xi} = \frac{\partial}{\partial \xi} \left(\sum_{i=1}^4 N_i x_i \right) = \sum_{i=1}^4 \frac{\partial N_i}{\partial \xi} x_i, \quad J_{12} = \frac{\partial y}{\partial \xi} = \frac{\partial}{\partial \xi} \left(\sum_{i=1}^4 N_i y_i \right) = \sum_{i=1}^4 \frac{\partial N_i}{\partial \xi} y_i,$$

$$J_{21} = \frac{\partial x}{\partial \eta} = \frac{\partial}{\partial \eta} \left(\sum_{i=1}^4 N_i x_i \right) = \sum_{i=1}^4 \frac{\partial N_i}{\partial \eta} x_i, \quad J_{22} = \frac{\partial y}{\partial \eta} = \frac{\partial}{\partial \eta} \left(\sum_{i=1}^4 N_i y_i \right) = \sum_{i=1}^4 \frac{\partial N_i}{\partial \eta} y_i$$

数值积分实施 (4/4)

```
!C
!C +-----+
!C | AREA |
!C +-----+
!C===
      AREA= 0.d0
      do jp= 1, 2
      do ip= 1, 2
        AREA= AREA + dabs(DETJ(ip,jp)) * W(ip) * W(jp)
      enddo
      enddo
```

```
!C
!C-- ANALYTICAL SOLUTION
      XA2= X(2) - X(1)
      YA2= Y(2) - Y(1)
      XA3= X(3) - X(1)
      YA3= Y(3) - Y(1)
      XA4= X(4) - X(1)
      YA4= Y(4) - Y(1)
```

$$I = \int_{-1}^{+1} \int_{-1}^{+1} f(\xi, \eta) d\xi d\eta = \sum_{i=1}^m \sum_{j=1}^n [W_i \cdot W_j \cdot f(\xi_i, \eta_j)]$$

```
      AREAa= 0.50d0 * (dabs(XA2*YA3-YA2*XA3) +dabs(XA3*YA4-YA3*XA4))
!C===

      write (*,'(a,1pe16.6)') 'Gaussian quadrature', AREA
      write (*,'(a,1pe16.6)') 'analytical sol.', AREAa

      stop
      end
```